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Problem solutions for Seminar 4, 16–20 February 2009

(For practical reasons some of the solutions may include problem parts that are not on the problem list for this seminar.)

EMEA, 9.7.1 (= MA I, 10.9.1)

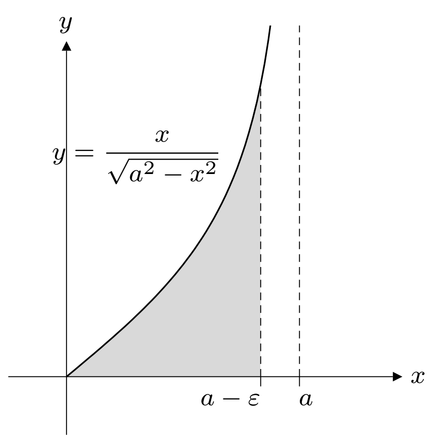
(a) $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b x^{-3}\,dx = \lim_{b\to\infty}\Big|_1^b -\frac{1}{2}x^{-2} = \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}\,.$

(b) The integral

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = \int_1^b x^{-1/2}\,dx = \Big|_1^b 2\sqrt{x} = 2\sqrt{b} - 2$$

does not converge to any limit as $b \to \infty$. Hence, the integral $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges.

(c) $\displaystyle\int_{-\infty}^0 e^x\,dx = \lim_{a\to-\infty}\int_a^0 e^x\,dx = \lim_{a\to-\infty}\Big|_a^0 e^x = \lim_{a\to-\infty}(1-e^a) = 1.$



Problem 9.7.1(d)

(d) If we introduce $u = \sqrt{a^2 - x^2}$ as a new variable we get $du = \dfrac{-x}{\sqrt{a^2 - x^2}}\,dx$ and

$$\int \frac{x}{\sqrt{a^2 - x^2}}\,dx = \int -du = -u + C = -\sqrt{a^2 - x^2} + C.$$

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Therefore,

$$\int_0^a \frac{x}{\sqrt{a^2-x^2}}\,dx = \lim_{\varepsilon\to 0^+}\int_0^{a-\varepsilon} \frac{x}{\sqrt{a^2-x^2}}\,dx = \lim_{\varepsilon\to 0^+} \Big|_0^{a-\varepsilon} -\sqrt{a^2-x^2}$$
$$= \lim_{\varepsilon\to 0^+}\big(-\sqrt{a^2-(a-\varepsilon)^2}+\sqrt{a^2}\,\big) = \sqrt{a^2} = a$$

(since $a > 0$). Note that we let ε tend to 0 from the right so that $a-\varepsilon$ tends to a from the left.

EMEA, 9.7.7 (= MA I, 10.9.7)

The integrand, $f(x) = \dfrac{1}{\sqrt{x+2}} + \dfrac{1}{\sqrt{3-x}}$, is defined only in the open interval $(-2, 3)$ and tends to ∞ at both ends of this interval. In order to show that the integral converges, we use the recipe in formula (9.7.3) on page 321 (formula (10.9.3) on page 362 in MA I): We split the interval at an arbitrary point, at $x = 0$, for instance, and then show that the integrals over $(-2, 0]$ and $[0, 3)$ both converge. The indefinite integral is

$$\int f(x)\,dx = F(x) + C, \quad \text{where} \quad F(x) = 2\sqrt{x+2} - 2\sqrt{3-x},$$

and it is clear that

$$\int_{-2}^0 f(x)\,dx = \lim_{a\to -2^+}(F(0) - F(a)) = F(0) - F(-2)$$

and

$$\int_0^3 f(x)\,dx = \lim_{b\to 3^-}(F(b) - F(0)) = F(3) - F(0)$$

exist. It follows that f is integrable over $(-2, 3)$ and

$$\int_{-2}^3 f(x)\,dx = \int_{-2}^0 f(x)\,dx + \int_0^3 f(x)\,dx = F(3) - F(-2) = 2\sqrt{5} + 2\sqrt{5} = 4\sqrt{5}.$$

Exam problem 21

(a) Using the rules

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C_1 \qquad (a \neq 0, \quad n \neq -1),$$

and

$$\int e^{ax} = \frac{e^{ax}}{a} + C_2 \qquad (a \neq 0),$$

we get

$$\int \big((2x-1)^2 + e^{2x-2}\big)\,dx = \frac{(2x-1)^3}{2\cdot 3} + \frac{e^{2x-2}}{2} + C.$$

(b) Here we could use polynomial division to simplify the fraction, but it is a little easier to use the substitution $u = x - 1$. Then $x = u + 1$, $dx = du$, and

$$\int \frac{x^2 - 2x}{x - 1}\,dx = \int \frac{(u+1)^2 - 2(u+1)}{u}\,du = \int \frac{u^2 - 1}{u}\,du$$
$$= \int \Big(u - \frac{1}{u}\Big)\,du = \frac{u^2}{2} - \ln|u| + C = \frac{(x-1)^2}{2} - \ln|x-1| + C.$$

(This can also be written as $\frac{1}{2}x^2 - x - \ln|x-1| + C_1$, with $C_1 = C + \frac{1}{2}$.)

(c) The innermost integral is

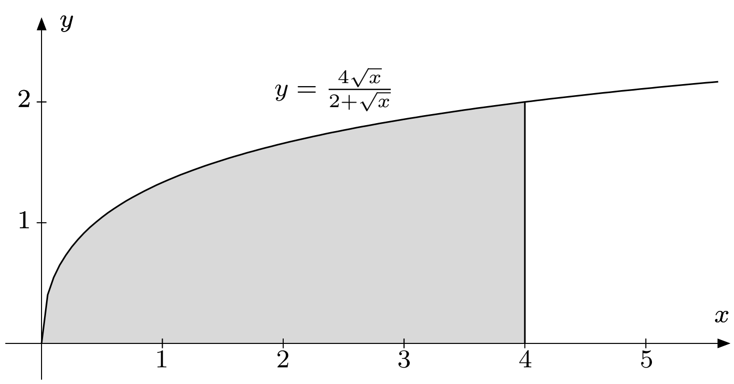
$$\int_1^2 \frac{1}{(x+y)^2}\,dx = \Big|_{x=1}^{x=2} -\frac{1}{x+y} = -\frac{1}{y+2} + \frac{1}{y+1},$$

and the double integral is therefore

$$\int_0^1 \Big(-\frac{1}{y+2} + \frac{1}{y+1}\Big)\,dy = \Big|_0^1 (-\ln(y+2) + \ln(y+1))$$
$$= (-\ln 3 + \ln 2) - (-\ln 2 + \ln 1) = 2\ln 2 - \ln 3 = \ln(4/3).$$

Exam problem 71

(a) The domain bounded by the curve $y = 4\sqrt{x}/(2 + \sqrt{x}\,)$, the x-axis, and the straight line $x = 4$ is shaded in the figure below.



Problem 71

The area of this domain is $A = \displaystyle\int_0^4 \frac{4\sqrt{x}}{2+\sqrt{x}}\,dx$. With the substitution $u = 2 + \sqrt{x}$ we get $x = (u-2)^2$, $dx = 2(u-2)\,du$, and

$$A = \int_{u=2}^{u=4} \frac{4(u-2)}{u}\,2(u-2)\,du = \int_2^4 \frac{8(u-2)^2}{u}\,du = \int_2^4 \Big(8u - 32 + \frac{32}{u}\Big)\,du$$
$$= \Big|_2^4 (4u^2 - 32u + 32\ln u) = (64 - 128 + 32\ln 4) - (16 - 64 + 32\ln 2)$$
$$= 32\ln 2 - 16 \quad (\approx 6.1807).$$

(b) L'Hôpital's rule yields

$$\lim_{x\to a}\frac{a^x - x^a}{x-a} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to a}\frac{a^x\ln a - ax^{a-1}}{1} = a^a(\ln a - 1).$$

Exam problem 77

(i) We first calculate the indefinite integral. Integration by parts gives

$$\begin{aligned}\int x(2+x)^{1/3}\,dx &= x\frac{3}{4}(2+x)^{4/3} - \frac{3}{4}\int (2+x)^{4/3}\,dx\\ &= x\frac{3}{4}(2+x)^{4/3} - \frac{9}{28}(2+x)^{7/3} + C\end{aligned}$$

The definite integral is then

$$\begin{aligned}\int_{-1}^{6} x(2+x)^{1/3}\,dx &= \Big|_{-1}^{6}\left(\frac{3x}{4}(2+x)^{4/3} - \frac{9}{28}(2+x)^{7/3}\right)\\ &= \frac{9}{2}8^{4/3} - \frac{9}{28}8^{7/3} - \left(-\frac{3}{4} - \frac{9}{28}\right) = \frac{447}{14} \approx 31.92,\end{aligned}$$

where we have used that $8^{1/3} = \sqrt[3]{8} = 2$.

Alternatively, we can use substitution and calculate as follows: Introduce $u = (2+x)^{1/3}$ as a new variable. That gives $x = u^3 - 2$, $dx = 3u^2\,du$, and

$$\begin{aligned}\int x(2+x)^{1/3}\,dx &= \int (u^3-2)u3u^2\,du = \int (3u^6 - 6u^3)\,du\\ &= \frac{3}{7}u^7 - \frac{6}{4}u^4 + C = \frac{3}{7}(2+x)^{7/3} - \frac{3}{2}(2+x)^{4/3} + C.\end{aligned}$$

(This is indeed equal to the indefinite integral we found above, although it does not look that way at first glance.)

We the calculate the definite integral as before. However, we can also use formula (2) on page 333 in EMEA (page 355 in MA I). That will give us

$$\int_{-1}^{6} x(2+x)^{1/3} = \int_{1}^{2}(3u^6 - 6u^3)\,du = \Big|_{1}^{2}\left(\frac{3}{7}u^7 - \frac{3}{2}u^4\right)$$

etc.

(ii) Here we use the substitution $z = \sqrt[3]{x} = x^{1/3}$, which gives $x = z^3$ and $dx = 3z^2\,dz$. The integral then becomes

$$\int e^{\sqrt[3]{x}}\,dx = \int e^z 3z^2\,dz = 3\int z^2e^z\,dz.$$

In order to find the last integral, we use integration by parts twice:

$$\begin{aligned}\int z^2e^z\,dz &= z^2e^z - \int 2ze^z\,dz = z^2e^z - \left(2ze^z - \int 2e^z\,dz\right)\\ &= z^2e^z - 2ze^z + \int 2e^z\,dz = z^2e^z - 2ze^z + 2e^z + C.\end{aligned}$$

Then

$$\int e^{\sqrt[3]{x}}\,dx = 3(z^2e^z - 2ze^z + 2e^z + C) = (3x^{2/3} - 6x^{1/3} + 6)e^{\sqrt[3]{x}} + C_1,$$

where $C_1 = 3C$.

Exam problem 97

(a) We have $\varphi(0) = \ln 1 - \ln 2 = -\ln 2$. Further,

$$\varphi'(x) = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+1)(x+2)} > 0$$

for all $x \geq 0$, so φ is strictly increasing. Finally,

$$\varphi(x) = \ln\frac{x+1}{x+2} = \ln\frac{1+1/x}{1+2/x} \to 0 \quad \text{as} \quad x \to \infty.$$

It follows that the range (Norwegian: "verdimengden") of φ is $V_\varphi = [-\ln 2, 0)$.

Note that it is not enough to observe that $\varphi(0) = -\ln 2$ and $\lim_{x\to\infty}\varphi(x) = 0$. The function value, $\varphi(x)$, might conceivably go both up and down as x runs from 0 to ∞, so that φ might take on values outside the interval $[-\ln 2, 0)$. But once we know that the function is increasing, this cannot happen.

(b) The function φ is strictly increasing throughout its domain (Norwegian: "definisjonsområde"), namely the interval $[0, \infty)$. Therefore it has an inverse φ^{-1} defined on the range $V_\varphi = [-\ln 2, 0)$ of φ. We can find a formula for the inverse in the following way:

$$\begin{aligned} y = \varphi^{-1}(x) &\iff \varphi(y) = x \iff \ln\frac{y+1}{y+2} = x \iff \frac{y+1}{y+2} = e^x \\ &\iff y + 1 = e^x(y+2) \iff y(1-e^x) = 2e^x - 1 \\ &\iff y = \frac{2e^x - 1}{1 - e^x}\,. \end{aligned}$$

(c) In order to find the inverse of the function φ', we first need to check whether φ' really has an inverse. We know that $\varphi'(x)$ is defined for $x > 0$. Because

$$\varphi''(x) = -\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} = \frac{(x+1)^2 - (x+2)^2}{(x+1)^2(x+2)^2} = \frac{-2x-3}{(x+1)^2(x+2)^2} < 0$$

for all $x > 0$, the function $\varphi'(x)$ is strictly decreasing throughout its domain, $D_{\varphi'} = (0, \infty)$. Since $\varphi'(x) \to 1/2$ as $x \to 0$ and $\varphi'(x) \to 0$ as $x \to \infty$, the range of φ' is $V_{\varphi'} = (0, 1/2)$. It follows that φ' has an inverse function, defined on $(0, 1/2)$.. Let $h = (\varphi')^{-1}$ be this inverse. Then h is defined on $(0, 1/2)$.

We have

$$\begin{aligned} y = h(x) &\iff \varphi'(y) = x \iff \frac{1}{(y+1)(y+2)} = x \iff (y+1)(y+2) = \frac{1}{x} \\ (*) \qquad &\iff y^2 + 3y + 2 - \frac{1}{x} = 0. \end{aligned}$$

It is tacitly understood here that y lies in $V_h = D_{\varphi'} = (0, \infty)$ and that x lies in $D_h = (0, 1/2)$. Equation $(*)$ is a quadratic equation for y. If we solve this equation we get

$$y = \frac{-3 \pm \sqrt{9 - 4(2 - \frac{1}{x})}}{2} = -\frac{3}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4}{x}} = -\frac{3}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{x}}\,.$$

Since y is supposed to be positive, it is clear that we have to choose the $+$ sign in front of the square root. Hence,

$$(\varphi')^{-1}(x) = h(x) = -\frac{3}{2} + \sqrt{\frac{1}{4} + \frac{1}{x}}\,.$$