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Problem solutions for Seminar 5, 23–27 February 2009

(For practical reasons some of the solutions may include problem parts that are not on the problem list for this seminar.)

EMEA, 9.9.2

(a) Here we have a separable equation:

$$\frac{dx}{dt} = \frac{e^{2t}}{x^2} = f(t)g(x),$$

where $f(t) = e^{2t}$ and $g(x) = 1/x^2$. We use the standard recipe for such equations. Separate:

$$x^2\,dx = e^{2t}\,dt.$$

Integrate:

$$\int x^2\,dx = \int e^{2t}\,dt,$$
$$\tfrac{1}{3}x^3 = \tfrac{1}{2}e^{2t} + C_1.$$

Solve for x:

$$x^3 = \tfrac{3}{2}e^{2t} + 3C_1 = \tfrac{3}{2}e^{2t} + C,$$

with $C = 3C_1$. Hence,

$$x = \sqrt[3]{\tfrac{3}{2}e^{2t} + C}.$$

(Note that you cannot let the constant wait and just stick it in "at the end". That would give a wrong answer, as in

$$\tfrac{1}{3}x^3 = \tfrac{1}{2}e^{2t}, \qquad x^3 = \tfrac{3}{2}e^{2t}, \qquad x = \sqrt[3]{\tfrac{3}{2}e^{2t}} + C.$$

This is not a solution of the given differential equation!)

There are no constant solutions of the equation because there is no value of x that makes $g(x) = 0$.

(b) This equation is also separable: $\dot{x} = e^{-t+x} = e^{-t}e^{x}$. The standard recipe gives

$$\int e^{-x}\,dx = \int e^{-t}\,dt\,,$$
$$-e^{-x} = -e^{-t} + C_1\,.$$

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Solve for x:

$$e^{-x} = e^{-t} + C, \quad \text{with} \quad C = -C_1 .$$

Hence,

$$-x = \ln(e^{-t} + C)\,, \quad \text{so} \quad x = -\ln(e^{-t} + C)\,.$$

(e) The equation $\dot{x} - 2x = -t$ is a linear differential equation of the form $\dot{x} + ax = b(t)$, with $a = -2$ and $b(t) = -t$. Hence, formula (9.9.5) on page 334 in EMEA (formula (1.4.5) on page 13 in FMEA) yields

$$\begin{aligned} x &= Ce^{-at} + e^{-at} \int e^{at} b(t)\, dt \\ &= Ce^{2t} + e^{2t} \int e^{-2t}(-t)\, dt = Ce^{2t} - e^{2t} \int te^{-2t}\, dt\,. \end{aligned}$$

Integration by parts gives

$$\int te^{-2t}\, dt = t(-\tfrac{1}{2})e^{-2t} + \tfrac{1}{2} \int e^{-2t}\, dt = (-\tfrac{1}{2}t - \tfrac{1}{4})e^{-2t}$$

and thus

$$x = Ce^{2t} - e^{2t}(-\tfrac{1}{2}t - \tfrac{1}{4})e^{-2t} = Ce^{2t} + \tfrac{1}{2}t + \tfrac{1}{4}\,.$$

Exam problem 111

This is a separable differential equation, and we get

$$\int \frac{3x^2}{(x^3+9)^{3/2}}\,dx = \int \ln t\,dt.$$

(Because we are looking for the solution through $(t,x) = (1,3)$, we need not worry about the single constant solution, $x \equiv -\sqrt[3]{9}$.) With $u = x^3+9$, we get $du = 3x^3\,dx$ and

$$\int \frac{3x^2}{(x^3+9)^{3/2}}\,dx = \int \frac{du}{u^{3/2}} = \int u^{-3/2}\,du$$
$$= -2u^{-1/2} + C_1 = -\frac{2}{\sqrt{x^3+9}} + C_1\,.$$

Integration by parts yields

$$\int \ln t = \int 1\cdot \ln t = t\ln t - t + C_2,$$

cf. Problem 9.5.3 in EMEA (10.6.3 in MA I). Alternatively, we could use the substitution $v = \ln t$, which yields $t = e^v$, $dt = e^v\,dv$, and

$$\int \ln t\,dt = \int ve^v\,dv = ve^v - \int 1\cdot e^v\,dv$$
$$= ve^v - e^v + C_2 = t\ln t - t + C_2\,.$$

(But note that we still do not avoid integration by parts.) Hence,

$$-\frac{2}{\sqrt{x^3+9}} = t\ln t - t + C\,, \quad \text{where} \quad C = C_2 - C_1\,.$$

The graph of the solution passes through $(t,x) = (1,3)$ if and only if

$$-\frac{2}{\sqrt{3^3+9}} = -1 + C\,,$$

i.e. $C = 1 - \frac{2}{6} = \frac{2}{3}$. This yields

$$-\frac{2}{\sqrt{x^3+9}} = t\ln t - t + \frac{2}{3}\,.$$

Let $\varphi(t) = t\ln t - t + 2/3$. Then,

$$-\frac{2}{\sqrt{x^3+9}} = \varphi(t) \implies \sqrt{x^3+9} = -\frac{2}{\varphi(t)} \implies x^3+9 = \frac{4}{\varphi(t)^2}$$
$$\implies x = \Big(\frac{4}{\varphi(t)^2} - 9\Big)^{1/3}.$$

Exam problem 127

(a) Let us first find the indefinite integral, using integration by parts:

$$\int \frac{\ln x^3}{x^2}\,dx = \int 3\ln x \cdot \frac{1}{x^2}\,dx = 3\ln x \cdot (-\frac{1}{x}) - \int \frac{3}{x}\cdot(-\frac{1}{x})\,dx$$
$$= -\frac{3\ln x}{x} + \int \frac{3}{x^2}\,dx = -\frac{3\ln x}{x} - \frac{3}{x} + C.$$

(We could have tried the substitution $u = \ln x$ instead. That would give $x = e^u$, $dx = e^u\,du$ and

$$\int \frac{\ln x^3}{x^2}\,dx = \int 3ue^{-2u}e^u\,du = \int 3ue^{-u}\,du,$$

and then we would have to use integration by parts here, too.)

The definite integral is then

$$\int_1^e \frac{\ln x^3}{x^2}\,dx = \Big|_1^e -\frac{3(\ln x + 1)}{x} = -\frac{3(1+1)}{e} + \frac{3(0+1)}{1} = 3 - 6e^{-1}.$$

(b) The equation can be written as

$$(**) \qquad \frac{dx}{dt} = (-1)\cdot\frac{x^2-25}{2x}, \qquad x > 5.$$

This is obviously a separable equation, and the standard procedure gives

$$\int \frac{2x\,dx}{x^2-25} = \int (-1)\,dt,$$
$$\ln(x^2-25) = -t + C,$$
$$x^2 - 25 = C_1 e^{-t}, \qquad (C_1 = e^C)$$
$$(1) \qquad x = \sqrt{25 + C_1 e^{-t}}.$$

Note that we have used the assumption $x > 5$. We want the solution that passes through $P = (t_0, x_0) = (0, 10)$. This leads to $10 = \sqrt{25 + C_1}$, so $C_1 = 75$ and

$$x = \sqrt{25 + 75e^{-t}} = 5\sqrt{1 + 3e^{-t}}.$$

The slope dx/dt of the curve at P is given by the differential equation (**):

$$\frac{dx}{dt} = -\frac{75}{20} = -\frac{15}{4}.$$

It is clear from (**) that dx/dt is always negative when $x > 5$, so all solutions are decreasing.

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Exam problem 120

(a) Formula (5.4.3) in FMEA ((1.4.3) in MA II) gives $x = Ce^{-2t} + 1$.

(b) $z = \dot{w}$ satisfies $\dot{z} + 2z = 2$, so $z = Ce^{-2t} + 1$ for a constant C. Hence,

$$w = \int z\,dt = \int (Ce^{-2t} + 1)\,dt = -\frac{1}{2}Ce^{-2t} + t + D,$$

with D as a new constant. $w(0) = 0$ gives $D = \frac{1}{2}C$, and $w(-\frac{1}{2}) = \frac{1}{2} - e$ gives $C = 2$. Hence $D = 1$. Thus the function we seek is

$$w(t) = -e^{-2t} + t + 1.$$

Exam problem 134

This is a linear differential equation with a variable right-hand side, and we shall use formula (5.4.4) on page 202 in FMEA (p. 199 in the first edition). This is the same as formula (1.4.5) on page 13 in MA II. The formula yields

$$\begin{aligned} x &= Ce^{-4t} + e^{-4t}\int e^{4t}\cdot 3e^{t}\,dt = Ce^{-4t} + e^{-4t}\int 3e^{5t}\,dt \\ &= Ce^{-4t} + \frac{3}{5}e^{-4t}e^{5t} = Ce^{-4t} + \frac{3}{5}e^{t}. \end{aligned}$$

The initial condition $x(0) = 1$ gives $1 = C + \frac{3}{5}$, so $C = \frac{2}{5}$. Hence

$$x = \frac{2}{5}e^{-4t} + \frac{3}{5}e^{t}.$$

(b) This is a separable equation.

$$\frac{dx}{dt} = \frac{e^{-3x}}{3 + \sqrt{t+8}} \iff \int e^{3x}\,dx = \int \frac{1}{3 + \sqrt{t+8}}\,dt.$$

Here $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C_1$. In the other integral we substitute $u = 3 + \sqrt{t+8}$. Then $u - 3 = \sqrt{t+8}$ and consequently $(u-3)^2 = t + 8$. Taking differentials, we get $2(u-3)\,du = dt$ and

$$\begin{aligned} \int \frac{1}{3+\sqrt{t+8}}\,dt &= \int \frac{2(u-3)}{u}\,du = 2\int du - 6\int \frac{1}{u}\,du \\ &= 2u - 6\ln|u| + C_2 = 2(3 + \sqrt{t+8}\,) - 6\ln(3 + \sqrt{t+8}) + C_2. \end{aligned}$$

This yields

$$\begin{aligned} \frac{1}{3}e^{3x} + C_1 &= 2(3 + \sqrt{t+8}\,) - 6\ln(3 + \sqrt{t+8}\,) + C_2 \\ &= 2\sqrt{t+8} - 6\ln(3 + \sqrt{t+8}\,) + C_2 + 6, \end{aligned}$$

so

$$e^{3x} = 6\sqrt{t+8} - 18\ln(3+\sqrt{t+8}\,) + C \quad \text{where} \quad C = 3(C_2 + 6 - C_1)$$

Hence

$$x = \frac{1}{3}\ln\bigl[6\sqrt{t+8} - 18\ln(3+\sqrt{t+8}\,) + C\bigr]$$

The initial condition $x(1) = 0$ gives $\frac{1}{3}\ln(18-18\ln 6+C) = 0$, so $18-18\ln 6+C = 1$, and finally $C = 18\ln 6 - 17$.