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Problem solutions for Seminar 6, 2–6 March 2009

(For practical reasons some of the solutions may include problem parts that are not on the problem list for this seminar.)

EMEA, 15.7.3 (= LA, 2.1.5)

Using the definitions of vector addition and multiplication of a vector by a real number, we get

$$3(x, y, z) + 5(-1, 2, 3) = (4, 1, 3) \iff (3x - 5, 3y + 10, 3z + 15) = (4, 1, 3)$$

Since two vectors are equal if and only if they are component-wise equal, this vector equation is equivalent to the equation system

$$3x - 5 = 4, \quad 3y + 10 = 1, \quad 3z + 15$$

with the obvious solution

$$x = 3, \quad y = -3, \quad z = -4.$$

EMEA, 15.7.8 (= LA, 2.2.4)

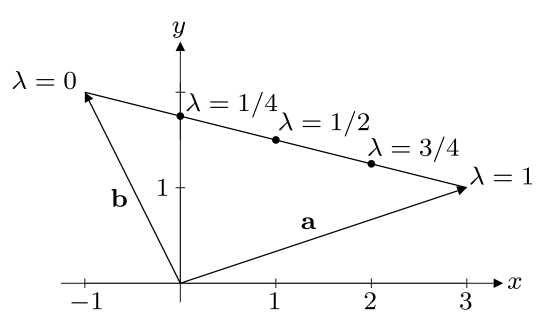
The dot product (inner product, scalar product; Norwegian: prikkproduktet, indreproduktet, skalarproduktet) is

$$(x, x - 1, 3) \cdot (x, x, 3x) = x^2 + (x - 1)x + 9x = 2x^2 + 8x = 2x(x + 4).$$

The two vectors are mutually orthogonal when their dot product equals 0, that is, for $x = 0$ and for $x = -4$.

EMEA, 15.8.2 (= LA, 2.3.2)

(a) *Note:* In the first two editions of EMEA, $\mathbf{x}$ was given as $\mathbf{x} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$, whereas in the third edition, and in LA, $\mathbf{x} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$. So if you have an old edition of EMEA, please read λ as $1 - \lambda$.



Problem 15.8.2

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(b) As λ runs from 0 to 1, $\mathbf{x}$ will trace out the line segment from $\mathbf{a}$ to $\mathbf{b}$ in EMEA (from $\mathbf{b}$ to $\mathbf{a}$ in LA). By the point-point formula for a straight line ("topunkts-formelen for en rett linje"), the straight line through $\mathbf{a} = (3, 1)$ and $\mathbf{b} = (-1, 2)$ has the equation

$$y - 1 = \frac{2-1}{-1-3}(x-3) = -\frac{1}{4}(x-3) \iff y = -\frac{1}{4}x + \frac{7}{4} \iff x + 4y = 7. \quad (*)$$

For every λ in $\mathbb{R}$ the point $(x, y) = (1-\lambda)\mathbf{a} + \lambda\mathbf{b} = (3-4\lambda, 1+\lambda)$ satisfies $(*)$, that is, the point lies on L. Conversely, if (x_0, y_0) is a point on L, then $x_0 = 7 - 4y_0$, and if we let $\lambda = y_0 - 1$, then $(1-\lambda)\mathbf{a} + \lambda\mathbf{b} = (3-4\lambda, 1+\lambda) = (7-4y_0, y_0) = (x_0, y_0)$.

EMEA 15.8.4 (= LA, 2.3.3)

(a) We get

$$x_1\mathbf{a} + x_2\mathbf{b} = (x_1, 2x_1, x_1) + (-3x_2, 0, -2x_2) = (x_1 - 3x_2, 2x_1, x_1 - 2x_2).$$

It follows that the vector equation $x_1\mathbf{a} + x_2\mathbf{b} = (5, 4, 4)$ is equivalent to the equation system

$$\begin{aligned} x_1 - 3x_2 &= 5 && (1) \\ 2x_1 \qquad &= 4 && (2) \\ x_1 - 2x_2 &= 4 && (3) \end{aligned}$$

From (2), $\underline{x_1 = 2}$, and if we substitute this value for x_1 in (1), we get $2 - 3x_2 = 5$, which yields $\underline{x_2 = -1}$. Inserting these values for x_1 and x_2, we find that they also satisfy equation (3).

Note: This check is important, because it might well happen that there were no values of x_1 and x_2 that would satisfy all the equations (1)–(3). We see an example of this in part (b).

(b) The vector equation $x_1\mathbf{a} + x_2\mathbf{b} = (-3, 6, 1)$ yields the equation system

$$\begin{aligned} x_1 - 3x_2 &= -3 && (1') \\ 2x_1 \qquad &= 6 && (2') \\ x_1 - 2x_2 &= 1 && (2') \end{aligned}$$

From $(2')$ we get $x_1 = 3$, and then $(1')$ yields $x_2 = 2$, but these values do not satisfy $(3')$. Hence there are no numbers x_1 and x_2 such that $x_1\mathbf{a} + x_2\mathbf{b} = (-3, 6, 1)$.

Problem 2

If the price vector is $\mathbf{p} = (p_1, p_2, p_3) = (4, 2, 5)$ and you are just able to afford the commodity vector $\mathbf{x}_0 = (6, 4, 3)$, then the amount of money at your disposal is

$$\mathbf{p} \cdot \mathbf{x}_0 = 4 \cdot 6 + 2 \cdot 4 + 5 \cdot 3 = 47$$

kroner (or doubloons or whatever the current unit of currency may be). Your budget constraint is therefore

$$p_1x_1 + p_2x_2 + p_3x_3 \leq 47,$$

meaning that if you want to buy a commodity vector $\mathbf{x} = (x_1, x_2, x_3)$, its total cost, $\mathbf{p} \cdot \mathbf{x}$, must not exceed 47.

Problem 3

(a) $\begin{pmatrix} 2 & -5 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} a & 1 & a+1 \\ 1 & 2 & 1 \\ 3 & 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 1 & 4 & 8 & 0 \\ 2 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

Problem 4

(a) $2\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} 7 & 0 \\ -1 & 11 \end{pmatrix}$ (b) $(\mathbf{A} - \mathbf{B})' = \begin{pmatrix} 3 & 0 \\ 1 & 5 \end{pmatrix}$

(c) $(\mathbf{C}'\mathbf{A}')\mathbf{B}' = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 5 & -1 \end{pmatrix}$

(d) $\mathbf{C}'(\mathbf{A}'\mathbf{B}') = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 5 & -1 \end{pmatrix}$. (Of course, we could have used the associative law, which says that $\mathbf{C}'(\mathbf{A}'\mathbf{B}') = (\mathbf{C}'\mathbf{A}')\mathbf{B}'$).

(e) $\mathbf{D}'\mathbf{D}'$ is not defined. (f) $\mathbf{D}'\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 10 & 13 \\ 5 & 13 & 17 \end{pmatrix}$

Problem 5

The equation is of the form $\dot{x} + a(t)x = b(t)$, i.e. linear with a variable coefficient. Formula (1.4.6) on page 15 in MA II or (5.4.6) in FMEA says that

$$\dot{x} + a(t)x = b(t) \iff x = e^{-\int a(t)\,dt}\Big(C + \int e^{\int a(t)\,dt} b(t)\,dt\Big) \qquad (*)$$

(This formula is not in EMEA.) For the given equation we have $a(t) = 2/t$ and $b(t) = e^t$, so formula $(*)$ yields

$$x = e^{-\int (2/t)\,dt}\Big(C + e^{\int (2/t)\,dt} e^t\,dt\Big).$$

Here $\int (2/t)\,dt = 2\ln|t| + C_1 = \ln t^2 + C_1$, and by choosing $C_1 = 0$ we get

$$x = e^{-\ln t^2}\Big(C + \int e^{\ln t^2} e^t\,dt\Big) = \frac{1}{t^2}\Big(C + \int t^2 e^t\,dt\Big)$$

Using integration by parts twice, we get

$$\int t^2 e^t\,dt = \cdots = t^2 e^t - 2te^t + 2e^t \quad (+\,C_2),$$

and so

$$x = \frac{C + (t^2 - 2t + 2)e^t}{t^2}.$$

The integral curve through $(t, x) = (1, 1)$ is obtained for the value of C that yields $x(1) = 1$, i.e. $1 = C + (1-2+2)e = C + e$. This gives $C = 1 - e$, and the solution is

$$x = \frac{1 - e + (t^2 - 2t + 2)e^t}{t^2}.$$

Exam problem 36

We are to study the function

$$f(x) = x - (\alpha+\beta)e^{-x} + \alpha e^{-2x} + \beta,$$

where α and β are constants and $\alpha > \beta > 0$.

(a)
$$f'(x) = 1 + (\alpha+\beta)e^{-x} - 2\alpha e^{-2x}$$
$$f''(x) = -(\alpha+\beta)e^{-x} + 4\alpha e^{-2x} = \frac{4\alpha - (\alpha+\beta)e^{x}}{e^{2x}}$$

(b) From the last expression for $f''(x)$, it follows that

$$f''(x) = 0 \iff (\alpha+\beta)e^{x} = 4\alpha \iff x = \bar{x} = \ln\frac{4\alpha}{\alpha+\beta}.$$

It also follows that $f''(x) > 0$ for $x < \bar{x}$ and $f''(x) < 0$ for $x > \bar{x}$, so $f''(x)$ changes sign around $x = \bar{x}$. Hence $\bar{x}$ is an inflection point for f, and it is the only inflection point because f'' has no other zeros.

Since $0 < \beta < \alpha$, we have

$$\frac{4\alpha}{\alpha+\beta} > \frac{4\alpha}{\alpha+\alpha} = 2,$$

and therefore

$$\bar{x} > \ln\frac{4\alpha}{\alpha+\beta} > \ln 2 > 0.$$

(c) The roots of the equation $2\alpha z^2 - (\alpha+\beta)z - 1 = 0$ are

$$z_1 = \frac{(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 8\alpha}}{4\alpha} \quad \text{and} \quad z_2 = \frac{(\alpha+\beta) - \sqrt{(\alpha+\beta)^2 + 8\alpha}}{4\alpha},$$

and because $\sqrt{(\alpha+\beta)^2 + 8\alpha} > \sqrt{(\alpha+\beta)^2} = \alpha+\beta$, we have $z_1 > 0$ and $z_2 < 0$. (We could also have used that $z_1 z_2 = -1/(2\alpha) < 0$, which shows that one root is positive and the other negative.)

(d) With $z = e^{-x_0}$, we see that

$$f'(x_0) = 0 \iff 1 + (\alpha+\beta)z - 2\alpha z^2 = 0.$$

We must have $z > 0$, and in part (c) we showed that the equation has exactly one positive solution, namely

$$z_1 = \frac{(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 8\alpha}}{4\alpha}.$$

It follows that $x_0 = -\ln z_1$ is the only stationary point of f.

We have

$$f'(x) = e^{-2x}[e^{2x} + (\alpha+\beta)e^{x} - 2\alpha],$$

where the expression in square brackets is a strictly increasing function of x. Since $[\cdots] = 0$ for $x = x_0$, we get

$$\begin{cases} f'(x) < 0 & \text{for } x < x_0, \\ f'(x) > 0 & \text{for } x > x_0. \end{cases}$$

Hence, f is strictly decreasing in $(-\infty, x_0]$ and strictly increasing in $[x_0, \infty)$, so x_0 must be a global minimum point of f.

(e) From the calculations in (d) we get

$$\begin{aligned} x_0 > 0 &\iff \ln z_1 < 0 \iff z_1 < 1 \iff \alpha + \beta + \sqrt{(\alpha+\beta)^2 + 8\alpha} < 4\alpha \\ &\iff \sqrt{(\alpha+\beta)^2 + 8\alpha} < 3\alpha - \beta \overset{(*)}{\iff} (\alpha+\beta)^2 < (3\alpha - \beta)^2 \\ &\iff \alpha^2 + 2\alpha\beta + \beta^2 + 8\alpha < 9\alpha^2 - 6\alpha\beta + \beta^2 \\ &\iff -8\alpha^2 + 8\alpha\beta + 8\alpha < 0 \iff -8\alpha(\alpha - \beta - 1) < 0 \\ &\iff \alpha - \beta - 1 > 0 \iff \underline{\alpha > \beta + 1}\,. \end{aligned}$$

The equivalence marked $(*)$ is valid because we know that $3\alpha - \beta > 0$ (otherwise only $\Longrightarrow$ would be valid).

The following is a simpler solution: We know that $f'(x)$ is positive everywhere to the right of x_0 and negative everywhere to the left. It follows that $x_0 > 0$ if and only if $f'(0) < 0$. Now $f'(0) = 1 + (\alpha+\beta)e^0 - 2\alpha e^0 = 1 + \beta - \alpha$, and this is negative if and only if $\alpha > \beta + 1$.