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Problem solutions for Seminar 7, 9–13 March 2009

Problem 1

Gaussian elimination yields

$$\begin{pmatrix} 1 & 1 & 0 & -2 & 2 \\ 0 & 2 & 1 & 1 & 3 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix} \begin{matrix} -1 \\ \big\downarrow \\ \leftarrow \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & -2 & 2 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix} \begin{matrix} \leftarrow \\ -1/2 \\ 1/3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1/2 & -5/2 & 1/2 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} \\ 1/2 \\ \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1/2 & -5/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} \leftarrow & \\ \leftarrow & \\ -1/2 & 5/2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The last matrix represents the equation system

$$\begin{aligned} x_1 \qquad - \tfrac{1}{2}x_3 \qquad &= \tfrac{1}{2} \\ x_2 + \tfrac{1}{2}x_3 \qquad &= \tfrac{3}{2} \\ x_4 &= 0 \end{aligned}$$

We can choose x_3 (or x_1 or x_2) arbitrarily, and the solutions are

$$x_1 = \frac{1}{2} + \frac{1}{2}s, \quad x_2 = \frac{3}{2} - \frac{1}{2}s, \quad x_3 = s, \quad x_4 = 0,$$

where s is any real number.

Exam problem 142

(a) Cofactor expansion along the first row yields

$$|\mathbf{A}| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 3 & -3 & -9 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ -3 & -9 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & -9 \end{vmatrix} + 4\begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix}$$
$$= 1(-15) - 3(-21) + 4(-12) = -15 + 63 - 48 = 0.$$

(b) We start Gaussian elimination:

$$\begin{pmatrix} 1 & 3 & 4 & b_1 \\ 2 & 2 & 1 & b_2 \\ 3 & -3 & -9 & b_3 \end{pmatrix} \begin{matrix} -2 \;\; -3 \\ \leftarrow\!\lrcorner \quad | \\ \longleftarrow\!\lrcorner \end{matrix} \sim \begin{pmatrix} 1 & 3 & 4 & b_1 \\ 0 & -4 & -7 & b_2 - 2b_1 \\ 0 & -12 & -21 & b_3 - 3b_1 \end{pmatrix} \begin{matrix} \\ -3 \\ \leftarrow\!\lrcorner \end{matrix}$$
$$\sim \begin{pmatrix} 1 & 3 & 4 & b_1 \\ 0 & -4 & -7 & b_2 - 2b_1 \\ 0 & 0 & 0 & 3b_1 - 3b_2 + b_3 \end{pmatrix}.$$

Thus the original system is equivalent to

$$\begin{aligned} x + 3y + 4z &= b_1 \\ -4y - 7z &= b_2 - 2b_1 \\ 0 &= 3b_1 - 3b_2 + b_3 \end{aligned}$$

It follows that for the system to have solutions, it is necessary that $3b_1 - 3b_2 + b_3 = 0$. This condition is also sufficient, because if it is satisfied, then the second equation yields y expressed in terms of z. We can then use the first equation to express x in terms of z. (An alternative is to use Gaussian elimination to the "bitter end", but it is unnecessary here because we have not been asked to find the solutions.)

Exam problem 69

(a) The determinant is

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & t \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & t+1 \end{vmatrix} = (t+1)\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = 2(t+1).$$

(We get the first equality by adding the second column to each of the other two columns. Thereafter we use cofactor expansion along the first row. An alternative is simply to use the definition of a determinant.)

(b) Gaussian elimination yields

$$
\begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 3 & 1 & -1 & 4 \end{pmatrix}
\begin{matrix} -1 \; -3 \\ \\ \\ \end{matrix}
\sim
\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -2 & -1 \\ 0 & 4 & -4 & -2 \end{pmatrix}
\begin{matrix} \\ 1/2 \\ \\ \end{matrix}
$$

$$
\sim
\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1/2 \\ 0 & 4 & -4 & -2 \end{pmatrix}
\begin{matrix} \\ 1 \quad -4 \\ \\ \end{matrix}
$$

$$
\sim
\begin{pmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

The last matrix shows that the system has the solution

$$x = 3/2, \quad y = a - 1/2, \quad z = a,$$

where a is arbitrary.

Exam problem 72

(a) Cofactor expansion along the second column gives

$$
\begin{vmatrix} a & 1 & 4 \\ 2 & 1 & a^2 \\ 1 & 0 & -3 \end{vmatrix}
= -\begin{vmatrix} 2 & a^2 \\ 1 & -3 \end{vmatrix}
+ \begin{vmatrix} a & 4 \\ 1 & -3 \end{vmatrix}
= -(-6 - a^2) + (-3a - 4) = a^2 - 3a + 2.
$$

(b) Gaussian elimination yields

$$
\begin{pmatrix} a & 1 & 4 & 2 \\ 2 & 1 & a^2 & 2 \\ 1 & 0 & -3 & a \end{pmatrix}
\sim
\begin{pmatrix} 1 & 0 & -3 & a \\ 2 & 1 & a^2 & 2 \\ a & 1 & 4 & 2 \end{pmatrix}
\begin{matrix} -2 \quad -a \\ \\ \\ \end{matrix}
$$

$$
\sim
\begin{pmatrix} 1 & 0 & -3 & a \\ 0 & 1 & a^2+6 & -2a+2 \\ 0 & 1 & 3a+4 & -a^2+2 \end{pmatrix}
\begin{matrix} \\ -1 \\ \\ \end{matrix}
$$

$$
\sim
\begin{pmatrix} 1 & 0 & -3 & a \\ 0 & 1 & a^2+6 & -2a+2 \\ 0 & 0 & -a^2+3a-2 & -a^2+2a \end{pmatrix}
$$

We can see from this that the system has a unique solution if and only if $-a^2 + 3a - 2 \neq 0$, that is, if and only if $a \neq 1$ and $a \neq 2$.

If $a = 2$, we get an infinite number of solutions, and if $a = 1$, there are no solutions at all.

(c) If we replace the right-hand sides with b_1, b_2, and b_3, and then perform the same operations as in part (b), we get an equation system with the augmented (or extended) coefficient matrix (Norwegian: "den utvidede koeffisientmatrisen")

$$\begin{pmatrix} 1 & 0 & -3 & b_3 \\ 0 & 1 & a^2+6 & b_2-2b_3 \\ 0 & 0 & -a^2+3a-2 & b_1-b_2+(2-a)b_3 \end{pmatrix}.$$

The system has an infinite number of solutions if and only if all elements in the last row are 0, that is, if and only if (i) $a = 1$ or $a = 2$, and (ii) $b_1 - b_2 + (2-a)b_3 = 0$.

(d) By the multiplication theorem for determinants, $|\mathbf{B}^3| = |\mathbf{B}|^3$, and since $\mathbf{B}$ is a 3×3 matrix, we have $|-\mathbf{B}| = (-1)^3|\mathbf{B}| = -|\mathbf{B}|$. Since $\mathbf{B}^3 = -\mathbf{B}$, it follows that $|\mathbf{B}|^3 = -|\mathbf{B}|$, which gives

$$|\mathbf{B}|(|\mathbf{B}|^2 + 1) = 0.$$

Therefore $|\mathbf{B}| = 0$, and so $\mathbf{B}$ cannot have an inverse.

Exam problem 41

(a)
$$f_1' = e^{x+y} + e^{x-y} - \tfrac{3}{2}, \qquad f_2' = e^{x+y} - e^{x-y} - \tfrac{1}{2},$$
$$f_{11}'' = e^{x+y} + e^{x-y}, \qquad f_{12}'' = e^{x+y} - e^{x-y}, \qquad f_{22}'' = e^{x+y} + e^{x-y}.$$

(b) We search for stationary points of f:

$$f_1'(x,y) = 0 \iff e^{x+y} + e^{x-y} = \tfrac{3}{2} \tag{1}$$
$$f_2'(x,y) = 0 \iff e^{x+y} - e^{x-y} = \tfrac{1}{2} \tag{2}$$

Adding equations (1) and (2) yields

$$2e^{x+y} = 2 \iff e^{x+y} = 1 \iff x + y = 0 \iff y = -x.$$

Setting $y = -x$ in (1) yields

$$1 + e^{2x} = \tfrac{3}{2} \iff e^{2x} = \tfrac{1}{2} \iff 2x = \ln(\tfrac{1}{2}) = -\ln 2 \iff x = -\tfrac{1}{2}\ln 2.$$

Hence, the only stationary point of f is $(x_0, y_0) = \left(-\tfrac{1}{2}\ln 2, \tfrac{1}{2}\ln 2\right)$.

From the results in (a) we can see that $f_{11}'' > 0$, $f_{22}'' > 0$, and

$$\begin{aligned} f_{11}''f_{22}'' - (f_{12}'')^2 &= (e^{x+y} + e^{x-y})^2 - (e^{x+y} - e^{x-y})^2 \\ &= e^{2x+2y} + 2e^{2x} + e^{2x-2y} - (e^{2x+2y} - 2e^{2x} + e^{2x-2y}) = 4e^{2x} > 0, \end{aligned}$$

for all x and y, so by Theorem 13.1.2 (13.1.1 in MA I), (x_0, y_0) is a global minimum point for f.