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Problem solutions for Seminar 8, 23–27 March 2009

(For practical reasons some of the solutions may include problem parts that are not on the problem list for this seminar.)

EMEA, 16.4.6 (= LA, 5.4.4)

This problem is an exercise in using some of the rules in Theorem 16.4.1 in EMEA (LA: setning 5.1).

(a) This determinant is zero because the second column equals 2 times the first column. We could also have used that the sum of the first and the second row equals the third row.

(b) By adding the second column to the third, we get

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = 0.$$

The last equality follows because the third column is proportional to the first.

(c) There is a common factor $x - y$ in the first row, so we get

$$\begin{vmatrix} x-y & x-y & x^2-y^2 \\ 1 & 1 & x+y \\ y & 1 & x \end{vmatrix} = (x-y)\begin{vmatrix} 1 & 1 & x+y \\ 1 & 1 & x+y \\ y & 1 & x \end{vmatrix} = 0,$$

since the new determinant has two equal rows.

EMEA, 16.4.10 (= LA, 5.4.8)

(a) The multiplication rule for determinants, part G of Theorem 16.4.1 on page 602 in EMEA (part 8 of Theorem 5.1 on page 98 i LA), shows that we must have

$$|\mathbf{A}|^2 = |\mathbf{A}^2| = |\mathbf{I}| = 1,$$

and therefore $|\mathbf{A}|$ equals 1 or -1.

(b) Direct matrix multiplication shows that the square of each of the two matrices is $\mathbf{I}_2$.

(c) We have

$$(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}) = \mathbf{II} - \mathbf{AI} + \mathbf{IA} - \mathbf{AA} = \mathbf{I} - \mathbf{A} + \mathbf{A} - \mathbf{A}^2 = \mathbf{I} - \mathbf{A}^2,$$

and this expression equals $\mathbf{0}$ if and only if $\mathbf{A}^2 = \mathbf{I}$.

Exam problem 5

(a) Expansion along the first row gives

$$|\mathbf{A}| = \begin{vmatrix} a & b & 0 \\ -b & a & b \\ 0 & -b & a \end{vmatrix} = a \begin{vmatrix} a & b \\ -b & a \end{vmatrix} - b \begin{vmatrix} -b & b \\ 0 & a \end{vmatrix} = a(a^2+b^2) - b(-ab) = a^3 + 2ab^2.$$

Matrix multiplication gives

$$\mathbf{AA} = \begin{pmatrix} a^2-b^2 & 2ab & b^2 \\ -2ab & a^2-2b^2 & 2ab \\ b^2 & -2ab & a^2-b^2 \end{pmatrix}.$$

(b) We have

$$(\mathbf{C}'\mathbf{B}\mathbf{C})' = \mathbf{C}'\mathbf{B}'(\mathbf{C}')' = \mathbf{C}'(-\mathbf{B})\mathbf{C} = -\mathbf{C}'\mathbf{B}\mathbf{C}.$$

(c) Since $\mathbf{A}' = \begin{pmatrix} a & -b & 0 \\ b & a & -b \\ 0 & b & a \end{pmatrix}$, the matrix $\mathbf{A}$ is skew-symmetric, that is, $\mathbf{A}' = -\mathbf{A}$, if and only if $a = 0$.

Exam problem 95

(a) Cofactor expansion along the first column gives

$$|\mathbf{A}_3(t)| = \begin{vmatrix} 3-t & -4 & 2 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{vmatrix} = (3-t)(-t)^2 - 1 \begin{vmatrix} -4 & 2 \\ 1 & -t \end{vmatrix}$$
$$= t^2(3-t) - (4t-2) = -t^3 + 3t^2 - 4t + 2.$$

To find $|\mathbf{A}_4(t,a)|$ we expand along the last column:

$$|\mathbf{A}_4(t,a)| = \begin{vmatrix} 3-t & -4 & 2 & a \\ 1 & -t & 0 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{vmatrix} = -a \begin{vmatrix} 1 & -t & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{vmatrix} + (-t)|\mathbf{A}_3(t)|$$
$$= -a + (-t)(-t^3 + 3t^2 - 4t + 2) = t^4 - 3t^3 + 4t^2 - 2t - a.$$

(b) From the last two equations we get $x_3 = x_2 - b_3$ and $x_2 = x_1 - b_2$. Therefore $x_3 = x_1 - b_2 - b_3$. Inserted into the first equation this yields $2x_1 - 4x_1 + 4b_2 + 2x_1 - 2b_2 - 2b_3 = b_1$, or

$$(*) \qquad b_1 - 2b_2 + 2b_3 = 0.$$

Thus, $(*)$ is a necessary condition for the system to have solutions. On the other hand, if $(*)$ is satisfied, then we can choose arbitrary values for x_1, and with $x_2 = a_1 - b_2$, $x_3 = x_1 - b_2 - b = 0$, it follows that the system has solutions with one degree of freedom.

(Alternative method: Gaussian elimination. By means of elementary operations on the augmented coefficient matrix ("den utvidede koeffisientmatrisen") we get

$$\begin{pmatrix} 2 & -4 & 2 & b_1 \\ 1 & -1 & 0 & b_2 \\ 0 & 1 & -1 & b_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -\frac{1}{2}b_1 + 2b_2 \\ 0 & 1 & -1 & -\frac{1}{2}b_1 + b_2 \\ 0 & 0 & 0 & \frac{1}{2}b_1 - b_2 + b_3 \end{pmatrix}$$

The conclusion follows.)

(c) We shall use the fact that a matrix $\mathbf{A}$ has an inverse if and only if $|\mathbf{A}| \neq 0$. In particular, we must have $|\mathbf{P}| \neq 0$.

(i) $|\mathbf{P}^2| = |\mathbf{P}|^2 \neq 0$, so $\mathbf{P}^2$ does have an inverse.

(ii) The determinant of $\mathbf{P} + \mathbf{P} = 2\mathbf{P}$ is $|2\mathbf{P}| = 2^n|\mathbf{P}|$, if $\mathbf{P}$ is $n \times n$. Then $|\mathbf{P} + \mathbf{P}| \neq 0$, so $\mathbf{P} + \mathbf{P}$ has an inverse. (Alternative solution: $(2\mathbf{P})(\frac{1}{2}\mathbf{P}^{-1}) = 2 \cdot \frac{1}{2}\mathbf{P}\mathbf{P}^{-1} = \mathbf{I}$, so $\frac{1}{2}\mathbf{P}^{-1}$ is the inverse of $2\mathbf{P} = \mathbf{P} + \mathbf{P}$.)

(iii) $|\mathbf{P}'| = |\mathbf{P}| \neq 0$, so $\mathbf{P}'$ has an inverse. (It is also a well-known fact that $(\mathbf{P}')^{-1} = (\mathbf{P}^{-1})'$.)

(iv) The matrix $\mathbf{P} + \mathbf{P}'$ need not have an inverse even if $\mathbf{P}$ has and inverse. Let $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\mathbf{P} + \mathbf{P}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$. The matrix $\mathbf{P}$ has an inverse if the determinant $|\mathbf{P}| = ad - bc$ is different from 0. Now, $|\mathbf{P} + \mathbf{P}'| = 4ad - (b+c)^2$, so in order to get an example of an invertible matrix $\mathbf{P}$ such that $\mathbf{P} + \mathbf{P}'$ is not invertible, it suffices to find numbers a, b, c, and d such that $ad - bc \neq 0$ and $4ad - (b+c)^2 = 0$.

Example: Let $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Then $|\mathbf{P}| = 1 \neq 0$, but $|\mathbf{P} + \mathbf{P}'| = \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0$.

Exam problem 121

(a) With the Lagrangian $\mathcal{L}(x, y) = e^x y - \lambda[(x-1)^2 + y^2 - 12]$ we find the necessary first-order conditions,

$$\begin{array}{lll} (1) & \mathcal{L}'_x = & e^x y - 2\lambda(x-1) = 0 \\ (2) & \mathcal{L}'_y = & e^x - 2\lambda y = 0 \\ (3) & & (x-1)^2 + y^2 = 12 \end{array}$$

We can see from (2) that $y \neq 0$ and $\lambda = e^x/2y$. If we substitute this expression for λ in (1), we get

$$e^x y - 2\frac{e^x}{2y}(x-1) = 0, \quad \text{which yields} \quad y^2 = x - 1.$$

Substituting this expression for y in (3), we get $x^2 - x - 12 = 0$, so $x = -3$ or $x = 4$. Since $x = y^2 + 1 \geq 1$, only $x = 4$ is possible. Then $y^2 = 3$, so $y = \pm\sqrt{3}$. Hence, the only points that satisfy the necessary conditions, (1)–(3), are $(4, \pm\sqrt{3})$.

Since the function $f(x, y) = e^x y$ is continuous and the circle

$$S = \{ (x, y) : (x-1)^2 + y^2 = 12 \}$$

is a closed and bounded set, the extreme value theorem shows that $f(x, y)$ attains a maximum value over S. The function values at the points we have found are

$$f(4, \pm\sqrt{3}) = \pm e^4\sqrt{3},$$

and this shows that $x = 4$, $y = \sqrt{3}$ solves the problem, with $f_{\text{maks}} = e^4\sqrt{3}$ and $\lambda = e^x/2y = \frac{1}{6}e^4\sqrt{3}$.

(b) The change in the maximum value of f is $\Delta f_{\text{maks}} \approx \lambda \cdot 0.03 = e^4\sqrt{3}/200$. Thus the percentage change is

$$\frac{\Delta f_{\text{maks}}}{f_{\text{maks}}} \cdot 100\,\% \approx \frac{e^4\sqrt{3}/200}{e^4\sqrt{3}} \cdot 100\,\% = 0.5\,\%.$$

Problem 6

(a) Cofactor expansion along the first row yields

$$|\mathbf{A}_t| = \begin{vmatrix} 1 & t & 0 \\ -2 & -2 & -1 \\ 0 & 1 & t \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 1 & t \end{vmatrix} - t \begin{vmatrix} -2 & -1 \\ 0 & t \end{vmatrix} = -2t + 1 + 2t^2$$

The equation $2t^2 - 2t + 1 = 0$ has no solutions. (In fact, $2t^2 - 2t + 1 = t^2 + (t-1)^2 > 0$ for all t.) Hence, $|\mathbf{A}_t| \neq 0$ for all t, and it follows that $\mathbf{A}_t$ has an inverse for every value of t.

(b) Matrix multiplication yields

$$(\mathbf{A}_t)^2 = \mathbf{A}_t\mathbf{A}_t = \begin{pmatrix} 1-2t & -t & -t \\ 2 & 3-2t & 2-t \\ -2 & t-2 & t^2-1 \end{pmatrix}$$

$$(\mathbf{A}_t)^3 = (\mathbf{A}_t)^2\mathbf{A}_t = \begin{pmatrix} 1 & 2t-2t^2 & t-t^2 \\ 4t-4 & 5t-4 & -t^2+4t-3 \\ 2-2t & t^2-4t+3 & t^3-2t+2 \end{pmatrix}$$

If $(\mathbf{A}_t)^3 = \mathbf{I}_3$, then every element on the main diagonal must be equal to 1. In particular, $5t - 4 = 1$, and therefore $t = 1$. Using the expression for $(\mathbf{A}_t)^3$ that we found above, it is easy to see that $(\mathbf{A}_1)^3$ is indeed equal to $\mathbf{I}_3$. It is clear that $t = 1$ is the only solution.

(c) Since $\mathbf{I}_3 = (\mathbf{A}_1)^3 = \mathbf{A}_1(\mathbf{A}_1)^2$, we have

$$(\mathbf{A}_1)^{-1} = (\mathbf{A}_1)^2 = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

(d) Since $\mathbf{A}$ is a square matrix and $\mathbf{A}'\mathbf{A} = \mathbf{I}$, it follows that $\mathbf{A}' = \mathbf{A}^{-1}$, and therefore $\mathbf{A}\mathbf{A}' = \mathbf{I}$ as well.

To show that $\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}$ is the inverse of $\mathbf{A}'\mathbf{B}\mathbf{A}$, it suffices to prove that their product is the identity matrix. We get

$$\begin{aligned}(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})(\mathbf{A}'\mathbf{B}\mathbf{A}) &= \mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\mathbf{A}'\mathbf{B}\mathbf{A} = \mathbf{A}'\mathbf{B}^{-1}\mathbf{I}\mathbf{B}\mathbf{A} \\ &= \mathbf{A}'\mathbf{B}^{-1}\mathbf{B}\mathbf{A} = \mathbf{A}'\mathbf{I}\mathbf{A} = \mathbf{A}'\mathbf{A} = \mathbf{I}.\end{aligned}$$

Alternatively, we could use the formula $(\mathbf{C}_1\mathbf{C}_2\mathbf{C}_3)^{-1} = (\mathbf{C}_3)^{-1}(\mathbf{C}_2)^{-1}(\mathbf{C}_1)^{-1}$ for the inverse of a product of three matrices, together with the fact that $\mathbf{A}' = \mathbf{A}^{-1}$:

$$(\mathbf{A}'\mathbf{B}\mathbf{A})^{-1} = (\mathbf{A}^{-1}\mathbf{B}\mathbf{A})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}(\mathbf{A}^{-1})^{-1} = \mathbf{A}'\mathbf{B}^{-1}\mathbf{A}.$$