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Problem solutions for Seminar 9, 30 March–3 April 2009

Exam problem 31

(a) (i)

$$\begin{aligned} f(tx_1, tx_2) &= 5(tx_1)^4 + 6(tx_1)(tx_2)^3 = 5t^4x_1^4 + 6tx_1t^3x_2^3 \\ &= t^4(tx_1^4 + 6x_1x_2^3 = t^4 f(x_1, x_2), \end{aligned}$$

so f is homogeneous of degree 4.

(ii) If F were homogeneous of degree k, then

$$F(t, t, t) = F(t \cdot 1, t \cdot 1, t \cdot 1) = t^k F(1, 1, 1),$$

but $F(t, t, t) = e^{3t}$ and $t^k F(1, 1, 1) = t^k e^3$, which is not the same as e^{3t} for any k. (The equality would have to hold for some *constant* k, and for all $t > 0$.)

(iii) $G(tK, tL, tM, tN) = \cdots = G(K, L, M, N) = t^0 G(K, L, M, N)$, so G is homogeneous of degree 0.

(b)

$$\begin{aligned} x_1f_1'(x_1, x_2) + x_2f_2'(x_1, x_2) &= x_1(20x_1^3 + 6x_2^3) + x_2 18x_1x_2^2 \\ &= 20x_1^4 + 24x_1x_2^3 = 4f(x_1, x_2), \end{aligned}$$

in accordance with Euler's theorem.

Exam problem 57

(a) Since

$$f(tx, ty) = (ty)^3 + 3(tx)^2(ty) = t^3y^3 + 3t^3x^2y = t^3 f(x, y),$$

f is homogeneous of degree 3. It follows that the desired constant is $k = 3$, cf. Euler's theorem.

Of course, we could also calculate directly:

$$xf_1'(x, y) + yf_2'(x, y) = x \cdot 6xy + y(3y^2 + 3x^2) = 3y^3 + 9x^2y = 3f(x, y).$$

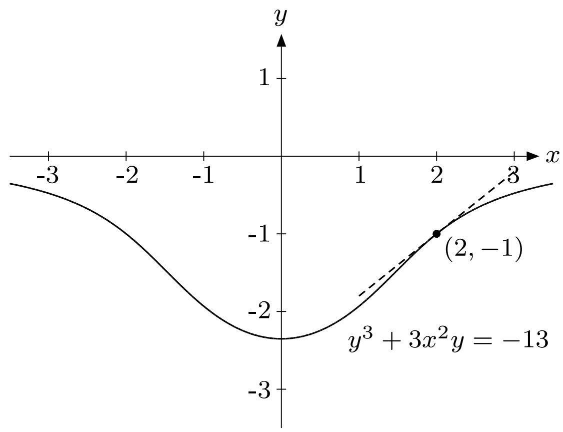
(b) For every value of x, the function $F(x, y) = y^3 + 3x^2y$ is strictly increasing with respect to y, with $F(x, y) \to -\infty$ as $y \to -\infty$ and $F(x, y) \to \infty$ as $y \to \infty$. It follows that the equation $F(x, y) = -13$ defines y as a function of x over the entire real line.

Implicit differentiation gives

$$\begin{aligned} 3y^2y' + 6xy + 3x^2y' &= 0, \\ y' = -\frac{6xy}{3x^2 + 3y^2} &= -\frac{2xy}{x^2 + y^2}\,. \end{aligned}$$

This is the slope of the tangent to the curve at the point (x, y). With $(x, y) = (2, -1)$ we get $y' = 4/5$. Hence, the tangent to the curve at the point $(2, -1)$ is given by the equation

$$y - (-1) = \frac{4}{5}(x - 2), \qquad \text{that is,} \qquad y = \frac{4}{5}x - \frac{13}{5}\,.$$



Exam problem 57. The level curve $f(x, y) = -13$ together with its tangent at $(2, -1)$.

(c) From $y' = \dfrac{-2xy}{x^2 + y^2}$ we get

$$y'' = \frac{(-2y - 2xy')(x^2 + y^2) - (-2xy)(2x + 2yy')}{(x^2 + y^2)^2}\,.$$

At the point $(2, -1)$ we have $y' = 4/5$ and

$$y'' = \frac{\left(2 - \frac{16}{5}\right)(4 + 1) - 4\left(4 - \frac{8}{5}\right)}{(4 + 1)^2} = \frac{(10 - 16) - \left(16 - \frac{32}{5}\right)}{25} = -\frac{78}{125} < 0.$$

Thus, y is a *concave* function of x around this point.

(d) Since $y(y^2 + 3x^2) = -13$, we have $(x, y) \neq (0, 0)$, and therefore

$$y = -\frac{13}{y^2 + 3x^2} < 0.$$

This shows that all points on the curve lie below the x-axis.

In part (b) we showed that

$$y' = -\frac{2xy}{x^2 + y^2} = \left(\frac{-2y}{x^2 + y^2}\right) \cdot x.$$

Since $\dfrac{-2y}{x^2 + y^2} > 0$, we have

$$y' < 0 \text{ for } x < 0 \quad \text{and} \quad y' > 0 \text{ for } x > 0.$$

This means that y decreases when x increases in $(-\infty, 0]$, and increases when x increases in $[0, \infty)$. Hence, y attains its least value, $y_{\min}$, for $x = 0$, and so $(y_{\min})^3 + 0 = -13$, which yields $y_{\min} = \sqrt[3]{-13} = -\sqrt[3]{13}$.

(Alternatively we could try to solve the problem

$$\text{minimize } y \quad \text{subject to} \quad y^3 + 3x^2y = -13 \tag{$*$}$$

by Lagrange's method. The Lagrangian is

$$\mathcal{L}(x, y) = y - \lambda(y^3 + 3x^2y + 13),$$

and the equations $\mathcal{L}'_1(x, y) = \mathcal{L}'_2(x, y) = 0$ give

$$-6\lambda xy = 0, \tag{1}$$

$$1 - 3\lambda y^2 - 3\lambda x^2 = 0. \tag{2}$$

We can see from equation (2) that we must have $\lambda \neq 0$. Moreover, we showed above that $y < 0$. Hence, from (1) we get $x = 0$, and the constraint yields $y = -\sqrt[3]{13}$. This is the only *possible* solution of the problem ($*$). But it then remains to show that it really *is* a solution of the problem.)

Exam problem 58

(a) Cofactor expansion along the first row yields

$$\begin{vmatrix} -2 & 4 & -t \\ -3 & 1 & t \\ t-2 & -7 & 4 \end{vmatrix} = -2 \begin{vmatrix} 1 & t \\ -7 & 4 \end{vmatrix} - 4 \begin{vmatrix} -3 & t \\ t-2 & 4 \end{vmatrix} - t \begin{vmatrix} -3 & 1 \\ t-2 & -7 \end{vmatrix}$$

$$= -2(4 + 7t) - 4(-12 - t^2 + 2t) - t(21 - t + 2) = 5t^2 - 45t + 40.$$

(b) The determinant of the coefficient matrix for the equation system is precisely the determinant that we calculated in part (a). Hence,

$$\text{the system has a unique solution} \iff 5t^2 - 45t + 40 \neq 0 \iff t \neq 1 \text{ and } t \neq 8.$$

(Here we used the fact that $5t^2 - 45t + 40 = 5(t - 1)(t - 8)$.)

(c) With $t = 8$ and $y = 3$ we get the equations

$$\begin{aligned} -2x - 8z &= -8 \\ -3x + 8z &= -32 \\ 6x + 4z &= 44 \end{aligned}$$

The solution of this system is $x = 8$, $z = -1$. (We can find this solution by using only the first two equations, but we must remember to check that it satisfies the third equation as well.)

(d) We sant to show that $(\mathbf{I} + s\mathbf{B})(\mathbf{I} + \mathbf{B}) = \mathbf{I}$ for a suitable value of s. If we use the fact that $\mathbf{B}^2 = 3\mathbf{B}$, we get $(\mathbf{I} + s\mathbf{B})(\mathbf{I} + \mathbf{B}) = \mathbf{I} + s\mathbf{B} + \mathbf{B} + s\mathbf{B}^2 = \mathbf{I} + \mathbf{B} + 4s\mathbf{B}$. The last expression equals $\mathbf{I}$ if and only if $4s\mathbf{B} = -\mathbf{B}$. This condition is satisfied if $s = -1/4$, and it follows that $(\mathbf{I} + \mathbf{B})^{-1} = \mathbf{I} - \frac{1}{4}\mathbf{B}$.

Exam problem 90

(a) The derivative of f is

$$f'(x) = 4xe^{-x^2-a} + (2x^2+a)(-2x)e^{-x^2-a} = 4x(1-\tfrac{1}{2}a-x^2)e^{-x^2-a}\,.$$

The stationary points are where $f'(x)=0$, i.e., where $x=0$ or $x^2 = 1-\frac{1}{2}a$. The latter equation has solutions only if $a \le 2$. Hence:

For $a < 2$, there are three stationary points,

$$x_1 = 0, \quad x_2 = -\sqrt{1-a/2}\,, \quad x_3 = \sqrt{1-a/2}\,.$$

For $a \ge 2$, there is only one stationary point, namely $x_1 = 0$.

(b) $f(-x) = (2(-x)^2+a)e^{-(-x)^2-a} = (2x^2+a)e^{-x^2-a} = f(x)$ for all x, so the graph of f is symmetric about the y-axis.

Since $\lim_{x\to\infty} u/e^u = 0$, we get

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\Big(2\frac{x^2}{e^{x^2}} + \frac{a}{e^{x^2}}\Big)e^{-a} = 0.$$

Because $f(-x) = f(x)$ we also get $\lim_{x\to-\infty} f(x) = 0$.

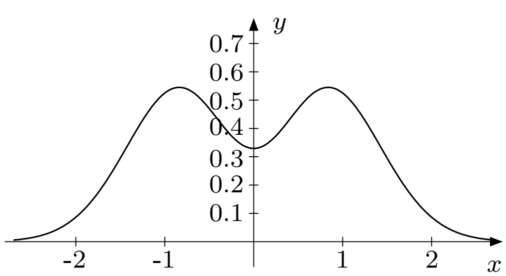
For $a \ge 2$ we have $f'(x) > 0$ if $x < 0$ and $f'(x) < 0$ if $x > 0$. Therefore $x = 0$ is a global maximum point for f, and $M(a) = f_{\max} = f(0) = ae^{-a}$.

For $a < 2$, $f'(x)$ has the same sign as $x(1-\frac{1}{2}a-x^2) = -x(x-x_2)(x-x_3)$. It follows that

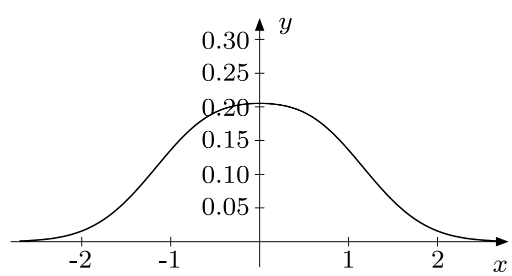
$$f'(x)\begin{cases} > 0 & \text{if } x < x_2 \\ < 0 & \text{if } x_2 < x < 0 \\ > 0 & \text{if } 0 < x < x_3 \\ < 0 & \text{if } x_3 < x \end{cases}$$

We have $x_2 = -x_3$, so by symmetry, $f(x_2) = f(x_3)$. Therefore $f(x)$ attains its maximum at x_2 and x_3, and we get $M(a) = f(x_3) = 2e^{-1-a/2}$.

The figures show the graph of f in the two cases $a = 0.6$ and $a = 2.5$. Note that the scales on the axes are different.



The graph of f with $a = 0.6$



The graph of f with $a = 2.5$

M is obviously continuous in each of the intervals $[0,2)$ and $(2,\infty)$. The one-sided limits of $M(a)$ at $a = 2$ are

$$\lim_{a\to 2^-} M(a) = \lim_{a\to 2^-} 2e^{-1-a/2} = 2e^{-2} \quad \text{and} \quad \lim_{a\to 2^+} M(a) = \lim_{a\to 2^+} ae^{-a} = 2e^{-2}.$$

Since the two one-sided limits are equal, we have $\lim_{a\to 2} M(a) = 2e^{-2} = M(2)$. It follows that M is continuous at $a = 2$ as well.

Since $M'(a) < 0$ both in $[0, 2)$ and in $(2, \infty)$, M is strictly decreasing in both $[0, 2]$ and $[2, \infty)$, and $M(a)$ has its greatest value when $a = 0$.

(d) We get

$$g_1'(x, y) = 4x(1 - \tfrac{1}{2}y - x^2)e^{-x^2-y}, \qquad g_2'(x, y) = (1 - y - 2x^2)e^{-x^2-y}$$

The stationary points are the solutions of the equations

$$(1)\ \ x(1 - \tfrac{1}{2}y - x^2) = 0 \quad \text{and} \quad (2)\ \ 1 - y - 2x^2 = 0$$

Equation (2) yields $y = 1 - 2x^2$. If we substitute this expression for y in (1), we get $x \cdot \frac{1}{2} = 0$. Hence, $(x, y) = (0, 1)$ is the only stationary point for g. The value of g at this stationary point is $g(0, 1) = e^{-1}$.

Now, $g(x, y)$ is the same as $f(x)$ with $a = y$. Thus, it follows from part (c) that for all $y \geq 0$ and all x we have $g(x, y) \leq M(y) \leq M(0) = 2e^{-1}$. Here, $M(0)$ is the maximum value of $f(x)$ when $a = 0$, so $g_{\max} = M(0) = f(\pm x_3) = f(\pm 1) = g(\pm 1, 0)$.

Exam problem 98

(a) The determinant of $\mathbf{A}_t$ is

$$|\mathbf{A}_t| = \begin{vmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -t & t \\ 2 & 1-t & t \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{(by subtracting column 3 from column 2)}$$

$$= \begin{vmatrix} 1 & -t \\ 2 & 1-t \end{vmatrix} = 1 - t + 2t = t + 1.$$

It follows that $\mathbf{A}_t$ has an inverse $\iff t \neq -1$.

(b) A straightforward calculation yields

$$\mathbf{I}_3 - \mathbf{B}\mathbf{A}_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 1 \\ 2 & 1 & t \end{pmatrix} = \begin{pmatrix} 0 & 0 & -t \\ 0 & 0 & -1 \\ -2 & -1 & 1-t \end{pmatrix}$$

The determinant of this matrix

$$|\mathbf{I}_3 - \mathbf{B}\mathbf{A}_t| = \begin{vmatrix} 0 & 0 & -t \\ 0 & 0 & -1 \\ -2 & -1 & 1-t \end{vmatrix} = 0.$$

(Use cofactor expansion along a row or a column, or note that columns number 1 and 2 are proportional, or add $-t$ times the second row to the first row to get a row with only zeros.) It follows that $\mathbf{I}_3 - \mathbf{B}\mathbf{A}_t$ does not have an inverse for any value of t.

Note: The determinant $|\mathbf{I}_3 - \mathbf{B}\mathbf{A}_t|$ does *not* equal $|\mathbf{I}_3| - |\mathbf{B}\mathbf{A}_t|$! The determinant of a sum or difference of two matrices usually does not equal the sum or difference of the two determinants.

The equation

$$\mathbf{B} + \mathbf{X}\mathbf{A}_1^{-1} = \mathbf{A}_1^{-1} \tag{1}$$

is equivalent to

$$\mathbf{X}\mathbf{A}_1^{-1} = \mathbf{A}_1^{-1} - \mathbf{B}. \tag{2}$$

Multiplying from the right by $\mathbf{A}_1$ yields

$$\mathbf{X} = \mathbf{I}_3 - \mathbf{B}\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix} \tag{3}$$

(Let $t = 1$ in the expression for $\mathbf{I}_3 - \mathbf{B}\mathbf{A}_t$ that we found above.)

Note that we are not finished yet. We have shown that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3), so we know that *if* $\mathbf{X}$ is a solution of (1), *then* $\mathbf{X}$ must be the matrix that we found in (3). But we do not know for sure that (1) really has a solution.

One way to find out is to check the answer, that is, inserting the matrix $\mathbf{X}$ that we have found into (1). (It turns out that $\mathbf{X}$ is indeed a solution.)

Alternatively we can use the fact that $\mathbf{A}_1$ has an inverse, and right-multiply by $\mathbf{A}_1^{-1}$ on both sides of equation (3). We then get

$$(3) \Rightarrow \mathbf{X}\mathbf{A}_1^{-1} = (\mathbf{I}_3 - \mathbf{B}\mathbf{A}_1)\mathbf{A}_1^{-1} = \mathbf{A}_1^{-1} - \mathbf{B},$$

i.e. (3) $\Rightarrow$ (2). It follows that (3) $\Leftrightarrow$ (2), and then (3) $\Leftrightarrow$ (1) as well.

(c) It is clear that $\mathbf{Y} = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{pmatrix}$ must be a 2×3 matrix such that

$$\mathbf{Y} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{pmatrix}.$$

The matrix equation

$$\begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{pmatrix},$$

leads to the following equations:

$$\begin{aligned} y_{11} &= 1, & 2y_{11} + y_{12} &= 2, & -3y_{11} + y_{13} &= -1, \\ y_{21} &= -1, & 2y_{21} + y_{22} &= 0, & -3y_{21} + y_{23} &= 4, \end{aligned}$$

We get

$$y_{11} = 1, \quad y_{12} = 2 - 2y_{11} = 0, \quad y_{13} = 3y_{11} + 1 = 2,$$

and in a similar fashion $y_{21} = -1$, $y_{22} = 2$ and $y_{23} = 1$. Thus the matrix $\mathbf{Y}$ is

$$\mathbf{Y} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

Alternatively we could have found the inverse of the matrix $\mathbf{C} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and used

$$\mathbf{YC} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{pmatrix} \iff \mathbf{Y} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{pmatrix} \mathbf{C}^{-1}.$$

$\mathbf{C}$ does have an inverse, because $|\mathbf{C}| = -1 \neq 0$, and a little work shows that

$$\mathbf{C}^{-1} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solutions of the extra problems:

Exam problem 66

(a) The determinant of the equation system is

$$\begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & a \\ -1 & a & -4 \end{vmatrix} = -a^2 - 7a + 18 = -(a+9)(a-2).$$

We can tell from this that the system has exactly one solution when $a \neq -9$ and $a \neq 2$.

If we let $a = -9$ and use Gaussian elimination, we get

$$\begin{pmatrix} 1 & 1 & -2 & -2 \\ 3 & -1 & -9 & -3 \\ -1 & -9 & -4 & 8 \end{pmatrix} \begin{matrix} -3 & 1 \\ \leftarrow & \\ & \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 1 & -2 & -2 \\ 0 & -4 & -3 & 3 \\ 0 & -8 & -6 & 6 \end{pmatrix} \begin{matrix} \\ -2 & -1/4 \\ \leftarrow & \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3/4 & -3/4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and this system obviously has solutions with one degree of freedom.

Similarly, with $a = 2$ we get

$$\begin{pmatrix} 1 & 1 & -2 & 9 \\ 3 & -1 & 2 & -3 \\ -1 & 2 & -4 & 8 \end{pmatrix} \begin{matrix} -3 & 1 \\ \hookleftarrow & \\ \longleftarrow & \end{matrix} \sim \begin{pmatrix} 1 & 1 & -2 & 9 \\ 0 & -4 & 8 & -30 \\ 0 & 3 & -6 & 17 \end{pmatrix} \begin{matrix} \\ 3/4 \\ \hookleftarrow \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -2 & 9 \\ 0 & -4 & 8 & -30 \\ 0 & 0 & 0 & -11/2 \end{pmatrix},$$

which shows that in this case the system is inconsistent ("selvmotsigende"), i.e. it has no solutions at all.

Exam problem 118

(a) Introduce $u = 1 + e^{\sqrt{x}}$ as a new variable. Then $u > 0$ and $du = \dfrac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}}\, dx$, so the integral equals $\displaystyle\int \frac{2\,du}{u} = 2 \ln u + C = 2\ln(1 + e^{\sqrt{x}}) + C$.

(b) Use integration by parts with $f(x) = \ln x$, $g'(x) = \sqrt{x}$. Formula (9.5.1) in EMEA (formula (10.6.1) in MA I) yields

$$\int_1^{e^2} \sqrt{x} \ln x\, dx = \Big|_1^{e^2} \frac{2}{3} x^{3/2} \ln x - \int_1^{e^2} \frac{2}{3} x^{3/2} \cdot \frac{1}{x}\, dx$$
$$= \Big|_1^{e^2} \frac{2}{3} x^{\frac{2}{3}} \ln x - \Big|_1^{e^2} \frac{4}{9} \cdot x^{3/2} = \frac{8}{9} e^3 + \frac{4}{9}\,.$$