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Problem solutions for Seminar 10, 15–20 April

EMEA, 12.8.7 (= 12.7.7 in the 1st ed. = 12.3.6 in MA I)

We shall use formula (3) on page 445 (page 444 in the 1st. ed.; formula (4) on page 434 in MA I) to find an equation for the tangent plane.

(a) Here, $\partial z/\partial x = 2x$ and $\partial z/\partial y = 2y$. At the point $(1, 2, 5)$, we get $\partial z/\partial y = 2$ and $\partial z/\partial x = 4$, so the tangent plane at this point has the equation

$$z - 5 = 2(x - 1) + 4(y - 5) \iff z = 2x + 4y - 5.$$

(b) From $z = (y - x^2)(y - 2x^2) = y^2 - 3x^2y + 2x^4$ we get $\partial z/\partial x = -6xy + 8x^3$ and $\partial z/\partial y = 2y - 3x^2$. Thus, at $(1, 3, 2)$ we have $\partial z/\partial x = -10$ and $\partial z/\partial y = 3$. The tangent plane is given by the equation

$$z - 2 = -10(x - 1) + 3(y - 3) \iff z = -10x + 3y + 3.$$

EMEA, 12.9.5 (= MA I, 12.4.6)

The equation $d(Ue^U) = d(x\sqrt{y})$ implies

$$e^U\,dU + Ue^U\,dU = \sqrt{y}\,dx + \frac{x}{2\sqrt{y}}\,dy\,.$$

Solving for dU yields

$$dU = \frac{\sqrt{y}}{e^U + Ue^U}\,dx + \frac{x}{(e^U + Ue^U)2\sqrt{y}}\,dy\,.$$

EMEA, 15.9.2 ((a),(b),(c) = LA, 2.6.2(a),(c),(d))

(a) The line L has the parameter representation

$$x_1 = -t + 2, \quad x_2 = 2t - 1, \quad x_3 = t + 3.$$

If we let $t = 0$, we get precisely the point $\mathbf{a} = (2, -1, 3)$ on L.

If the point $(1, 1, 1)$ were to lie on L, there must exist a t such that

$$-t + 2 = 1, \quad 2t - 1 = 1, \quad t + 3 = 1.$$

But the first of these equations gives $t = 1$, which is not a solution of the last equation. Thus $(1, 1, 1)$ cannot lie on L.

(b) To an arbitrary t there corresponds the point

$$\mathbf{x} = (2, -1, 3) + (-t, 2t, t) = \mathbf{a} + t\mathbf{v},$$

where $\mathbf{v} = (-1, 2, 1)$. Thus, the vector $\mathbf{v}$ is a direction vector for the line L.

(c) The plane $\mathcal{P}$ that passes through $\mathbf{a}$ and is perpendicular to L, has $\mathbf{v}$ as a normal vector. (Every direction vector for L is a normal vector for $\mathcal{P}$ and vice versa.) Therefore $\mathcal{P}$ is given by the equation

$$\mathbf{v} \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{a},$$

which gives

$$-x_1 + 2x_2 + x_3 = -1.$$

(d) The point of intersection between L and the plane $3x_1 + 5x_2 - x_3 = 6$ is given by that parameter value t for which the point $(-t + 2, 2t - 1, t + 3)$ on L satisfies the equation for the plane, that is

$$3(-t + 2) + 5(2t - 1) - (t + 3) = 6.$$

This equation gives $t = 4/3$, and we get the point $(2/3, 5/3, 13/3)$.

EMEA, 15.9.5 (= LA, 2.6.4)

(a) With the components of $\mathbf{a}$ as x, y, and z, respectively, the equation for the plane is satisfied.

(b) Let $\mathbf{n} = (-1, 2, 3)$. Then $\mathbf{n}$ is a normal vector to the plane. In fact, the equation for the plane can be written as $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a}$, where $\mathbf{x} = (x, y, z)$. Then the normal to the plane at the point $\mathbf{a}$ is the straight line through $\mathbf{a}$ with $\mathbf{n}$ as a direction vector.

A parametric representation of this line can be given as $\mathbf{x} = \mathbf{a} + t\mathbf{n}$, i.e.,

$$(x, y, z) = (-2, 1, -1) + t(-1, 2, 3)$$

or, equivalently,

$$x = -2 - t, \quad y = 1 + 2t, \quad z = -1 + 3t.$$

Exam problem 44

(a) Using the chain rule, we get

$$\begin{aligned} f'_x(x, y) &= ye^{-x/y} + xye^{-x/y} \cdot \Big(-\frac{\partial}{\partial x}\Big(\frac{x}{y}\Big)\Big) \\ &= ye^{-x/y} + xye^{-x/y}\Big(-\frac{1}{y}\Big) = (y - x)e^{-x/y} \\ f'_y(x, y) &= xe^{-x/y} + xye^{-x/y} \cdot \Big(-\frac{\partial}{\partial y}\Big(\frac{x}{y}\Big)\Big) \\ &= xe^{-x/y} + xye^{-x/y} \cdot \frac{x}{y^2} = \Big(x + \frac{x^2}{y}\Big)e^{-x/y} \end{aligned}$$

(b) If we let $u = -x/y$, we get

$$\begin{aligned}\mathrm{El}_x f(x,y) &= \mathrm{El}_x x + \mathrm{El}_x y + \mathrm{El}_x e^u = 1 + 0 + \mathrm{El}_u e^u \cdot \mathrm{El}_x u \\ &= 1 + u(\mathrm{El}_x x - \mathrm{El}_x y) = 1 + u(1-0) = 1 + u = 1 - \frac{x}{y} \\ \mathrm{El}_y f(x,y) &= \mathrm{El}_y x + \mathrm{El}_y y + \mathrm{El}_u e^u \cdot \mathrm{El}_y u \\ &= 0 + 1 + u(0-1) = 1 - u = 1 + \frac{x}{y}\end{aligned}$$

Check:

$$\begin{aligned}\mathrm{El}_x f(x,y) &= \frac{x}{f(x,y)} \cdot f'_x(x,y) = \frac{x}{xye^{-x/y}}(y-x)e^{-x/y} = \frac{y-x}{y} = 1 - \frac{x}{y} \\ \mathrm{El}_y f(x,y) &= \frac{y}{f(x,y)} \cdot f'_y(x,y) = \frac{y}{xye^{-x/y}}\Big(x + \frac{x^2}{y}\Big)e^{-x/y} = 1 + \frac{x}{y}\end{aligned}$$

(c) The function f has first-order partial derivatives throughout its domain of definition, and this domain contains none of its boundary points. (The boundary of the domain consists of the nonnegative parts of the coordinate axes. Draw a picture!) Hence any maximum point of f must be a stationary point of f. But f has no stationary points, because $f'_y(x,y) = (x + x^2/y)e^{-x/y} > 0$ when $x > 0$ and $y > 0$. Therefore f has no maximum point, not even a local one.

We can also see directly that we can get as large function values as we like: For $t > 0$, $f(t,t) = t^2 \cdot e^{-1}$, which tends to ∞ as $t \to \infty$.

(d) We shall use Lagrange's method to maximize $f(x,y)$ subject to the constraint $x + y = c$. With the Lagrangian

$$\mathcal{L}(x,y) = f(x,y) - \lambda(x+y-c) = xye^{-x/y} - \lambda(x+y-c)$$

we get the first-order conditions

$$\begin{aligned}\mathcal{L}'_x(x,y) &= f'_x(x,y) - \lambda = 0, \\ \mathcal{L}'_y(x,y) &= f'_y(x,y) - \lambda = 0,\end{aligned}$$

which yield $f'_x(x,y) = f'_y(x,y)$. According to part (a), we must then have

$$(*) \qquad y - x = x + \frac{x^2}{y}, \quad \text{that is,} \quad (y-x)y = xy + x^2.$$

The constraint yields $y = c - x$, and if we substitute this expression for y in $(*)$ and "tidy up" the equation, we get

$$2x^2 - 4cx + c^2 = 0.$$

This quadratic equation has the solutions

$$x = \frac{4c \pm \sqrt{16c^2 - 8c^2}}{4} = \cdots = c \pm \frac{\sqrt{2}\,c}{2}.$$

Since we must have $0 < x < c$, only $x = x^* = (1 - \frac{1}{2}\sqrt{2})c$ is usable. The corresponding value of y is $y^* = c - x^* = \frac{1}{2}\sqrt{2}\,c$. Hence, the maximum value of $f(x, y)$ subject to the given constraint is

$$f(x^*, y^*) = x^* y^* e^{-x^*/y^*} = \frac{\sqrt{2} - 1}{2}\, e^{1-\sqrt{2}}\, c^2.$$

Exam problem 79

(a) Since the elasticity of a product is the sum of elasticities of each factor, we have

$$(*) \qquad \begin{aligned} \mathrm{El}_x\, e^{x-y} + \mathrm{El}_x \ln(x + z - 1) &= \mathrm{El}_x \sqrt{xy} \\ \mathrm{El}_x\, x^2 + \mathrm{El}_x\, y^3 + \mathrm{El}_x\, z &= 0. \end{aligned}$$

Using the rules for elasticities (see Section 5.13 in MA I or Problem 7.7.10 in EMEA), we get

$$\begin{aligned} \mathrm{El}_x\, e^{x-y} &= \mathrm{El}_x (e^x/e^y) = \mathrm{El}_x\, e^x - \mathrm{El}_x\, e^y = x - y\, \mathrm{El}_x\, y, \\ \mathrm{El}_x \ln(x + z - 1) &= \mathrm{El}_x \ln u = \mathrm{El}_u \ln u\, \mathrm{El}_x\, u = \frac{1}{\ln u} \frac{x\, \mathrm{El}_x\, x + z\, \mathrm{El}_x\, z}{x + z - 1} \\ &= \frac{1}{\ln(x + z - 1)} \frac{x + z\, \mathrm{El}_x\, z}{x + z - 1}, \qquad \text{with } u = x + z - 1, \\ \mathrm{El}_x \sqrt{xy} &= \mathrm{El}_x (xy)^{1/2} = \frac{1}{2}\, \mathrm{El}_x (xy) = \frac{1}{2}(1 + \mathrm{El}_x\, y), \\ \mathrm{El}_x\, x^2 &= 2, \qquad \mathrm{El}_x\, y^3 = 3\, \mathrm{El}_x\, y. \end{aligned}$$

Inserting this into the equation system $(*)$, we get

$$\begin{aligned} x - y\, \mathrm{El}_x\, y + \frac{x + z\, \mathrm{El}_x\, z}{(x + z - 1)\ln(x + z - 1)} &= \frac{1}{2} + \frac{1}{2}\, \mathrm{El}_x\, y \\ 2 + 3\, \mathrm{El}_x\, y + \mathrm{El}_x\, z &= 0. \end{aligned}$$

In particular, for $x = 1$, $y = 1$, and $z = e$ we have

$$\begin{aligned} 1 - \mathrm{El}_x\, y + \frac{1 + e\, \mathrm{El}_x\, z}{e} &= \frac{1}{2} + \frac{1}{2}\, \mathrm{El}_x\, y \\ 2 + 3\, \mathrm{El}_x\, y + \mathrm{El}_x\, z &= 0. \end{aligned}$$

This leads to the equation system

$$\begin{aligned} -\frac{3}{2}\, \mathrm{El}_x\, y + \mathrm{El}_x\, z &= -\frac{1}{2} - \frac{1}{e} \\ 3\, \mathrm{El}_x\, y + \mathrm{El}_x\, z &= -2, \end{aligned}$$

with the solution

$$\mathrm{El}_x\, y = \frac{2}{9e} - \frac{1}{3} \approx -0.25158, \qquad \mathrm{El}_x\, z = -1 - \frac{2}{3e} \approx -1.24525.$$

(b) If x increases from 1 to 1.1, i.e. by 10 %, then y will *decrease* by approximately 2.5 % and z will *decrease* by about 12.5 %.

(These percentages are only approximations to the real values. Numerical solution of the equation system in the problem yields $y \approx 0.977730$ and $z \approx 2.403546 \approx 0.884215e$ for $x = 1.1$. This corresponds to a reduction of approximately 2.2270 % for y and 11.5785 % for z.)

Exam problem 108

(a)
$$\begin{aligned}|\mathbf{A}| &= \begin{vmatrix} q & -1 & q-2 \\ 1 & -p & 2-p \\ 2 & -1 & 0 \end{vmatrix} \overset{(1)}{=} \begin{vmatrix} q-2 & -1 & q-2 \\ 1-2p & -p & 2-p \\ 0 & -1 & 0 \end{vmatrix} \\ &\overset{(2)}{=} (-1)(-1)\begin{vmatrix} q-2 & q-2 \\ 1-2p & 2-p \end{vmatrix} \\ &= (q-2)(2-p) - (q-2)(1-2p) = (q-2)(p+1).\end{aligned}$$

Equality $\overset{(1)}{=}$ comes from adding 2 times the second column to the first column, and we get $\overset{(2)}{=}$ by cofactor expansion along the last row.

Straightforward matrix multiplication gives

$$\mathbf{AE} = \begin{pmatrix} q & -1 & q-2 \\ 1 & -p & 2-p \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2q-3 & 2q-3 & 2q-3 \\ 3-2p & 3-2p & 3-2p \\ 1 & 1 & 1 \end{pmatrix}.$$

(All columns in the product matrix are equal, and that is because all columns in the matrix $\mathbf{E}$ are equal.)

Further,

$$\begin{aligned}|\mathbf{A}+\mathbf{E}| &= \begin{vmatrix} q+1 & 0 & q-1 \\ 2 & 1-p & 3-p \\ 3 & 0 & 1 \end{vmatrix} \overset{(3)}{=} (1-p)\begin{vmatrix} q+1 & q-1 \\ 3 & 1 \end{vmatrix} \\ &= (1-p)[(q+1) - 3(q-1)] = (1-p)(4-2q) = 2(1-p)(2-q),\end{aligned}$$

where we get equality $\overset{(3)}{=}$ by cofactor expansion along the second column.

(b) $\mathbf{A}+\mathbf{E}$ has an inverse $\iff |\mathbf{A}+\mathbf{E}| \neq 0 \iff p \neq 1$ and $q \neq 2$.

Since the columns of $\mathbf{E}$ are all equal, we have $|\mathbf{E}| = 0$. Then for all 3×3 matrices $\mathbf{B}$ we also have $|\mathbf{BE}| = |\mathbf{B}|\,|\mathbf{E}| = 0$. Therefore $\mathbf{BE}$ cannot have an inverse.

(c) The last equation gives $y = 2x$. Hence the system has a solution (unique solution) if and only if

$$(q-2)x = q-2 \tag{1}$$
$$(1-2p)x = 2-p \tag{2}$$

has a solution (unique solution). If $p = 1/2$, then (2) becomes $0 = 2 - p = 3/2$, which is impossible. If $p \neq 1/2$, then (2) yields a unique value for x, namely $x = (2-p)/(1-2p)$.

It remains to be seen if this value also satisfies (1), that is, $(q-2)(x-1)=0$. If $q=2$, there is no problem. If $q \neq 2$, then we must have $x=1$, and this implies $2-p=1-2p$, which is equivalent to $p=-1$.

Putting all this together, we have the following:

$$\begin{cases} p = 1/2: & \text{No solution.} \\ p \neq 1/2 \text{ and } q = 2: & \text{Unique solution.} \\ p \neq -1 \text{ and } q \neq 2: & \text{No solution.} \\ p = -1: & \text{Unique solution.} \end{cases}$$

Solutions of the extra problems:

Exam problem 62

(a) The determinant of $\mathbf{A}_a$ is

$$|\mathbf{A}_a| = \begin{vmatrix} 1 & -a & -a \\ -a & 1 & -a \\ -a & -a & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -a \\ -a & 1 \end{vmatrix} - (-a) \begin{vmatrix} -a & -a \\ -a & 1 \end{vmatrix} + (-a) \begin{vmatrix} -a & 1 \\ -a & -a \end{vmatrix}$$
$$= -2a^3 - 3a^2 + 1.$$

It is easy to see that $-2a^3 - 3a^2 + 1 = 0$ for $a = 1/2$. Hence, $a - 1/2$ is a factor in $-2a^3 - 3a^2 + 1$. Polynomial division gives

$$(-2a^3 - 3a^2 + 1) \div (a - 1/2) = -2a^2 - 4a - 2 = -2(a+1)^2,$$

so $|\mathbf{A}_a| = -2(a+1)^2(a-1/2)$, and

$$|\mathbf{A}_a| \neq 0 \iff a \neq -1 \text{ and } a \neq 1/2.$$

Thus, $\mathbf{A}_a$ has an inverse precisely when a is different from -1 and $1/2$.

Note: In English, division of a by b is usually written as $a \div b$ rather than $a : b$.

(b) Let $\mathbf{B} = k \begin{pmatrix} 1-a & a & a \\ a & 1-a & a \\ a & a & 1-a \end{pmatrix}$. The product of $\mathbf{A}_a$ and $\mathbf{B}$ is

$$\mathbf{A}_a\mathbf{B} = \begin{pmatrix} 1 & -a & -a \\ -a & 1 & -a \\ -a & -a & 1 \end{pmatrix} \cdot k \begin{pmatrix} 1-a & a & a \\ a & 1-a & a \\ a & a & 1-a \end{pmatrix}$$
$$= k \begin{pmatrix} 1-a-2a^2 & 0 & 0 \\ 0 & 1-a-2a^2 & 0 \\ 0 & 0 & 1-a-2a^2 \end{pmatrix}$$
$$= k(1-a-2a^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= k(1-a-2a^2)\,\mathbf{I}_3 = k(1+a)(1-2a)\,\mathbf{I}_3.$$

This shows that if we choose $k = 1/(1-a-2a^2)$, then $\mathbf{B}$ is the inverse of $\mathbf{A}_a$, and this works for all values of a except -1 and $1/2$.

(c) Let $\mathbf{x} = (x_1, x_2, x_3)'$. Then

$$\mathbf{A}_a^{-1}\mathbf{x} = k \begin{pmatrix} 1-a & a & a \\ a & 1-a & a \\ a & a & 1-a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} (1-a)x_1 + ax_2 + ax_3 \\ ax_1 + (1-a)x_2 + ax_3 \\ ax_1 + ax_2 + (1-a)x_3 \end{pmatrix}.$$

If $0 < a < 1/2$, then k is positive, and then all components of $\mathbf{A}_a^{-1}\mathbf{x}$ are positive if the components x_1, x_2, and x_3 of $\mathbf{x}$ are positive.

(Actually, it is sufficient to note that the elements of $\mathbf{A}_a^{-1}$ are positive when $a \in (0, 1/2)$. Then it follows directly from the definition of matrix multiplication that all components of $\mathbf{A}_a^{-1}\mathbf{x}$ will be positive when the components of $\mathbf{x}$ are positive. Note that if $a \in (1/2, 1)$, then all elements of $\mathbf{A}_a$ will be *negative*, since $k < 0$ for these values of a.)

Exam problem 14

The given differential equation

$$t\dot{x} + (2-t)x = e^{2t}, \quad t > 0 \tag{$*$}$$

is equivalent to

$$\dot{x} + a(t)x = b(t), \quad \text{where} \quad a(t) = \frac{2-t}{t} = \frac{2}{t} - 1, \quad b(t) = \frac{e^{2t}}{t}.$$

We choose an indefinite integral of $a(t)$:

$$A(t) = \int a(t)\,dt = 2\ln t - t.$$

The general solution of $(*)$ is then

$$\begin{aligned} x(t) &= e^{-A(t)}\left(\int e^{A(t)}b(t)\,dt + C\right) = \frac{e^t}{t^2}\left(\int t^2 e^{-t}\frac{e^{2t}}{t}\,dt + C\right) \\ &= \frac{e^t}{t^2}\left(\int te^t\,dt + C\right) = \frac{e^t}{t^2}(te^t - e^t + C) = \underline{\underline{\frac{e^{2t}(t-1)}{t^2} + \frac{Ce^t}{t^2}}}. \end{aligned}$$

In particular, $x(1) = Ce$, so the particular solution with $x(1) = 0$ is

$$x(t) = \frac{e^{2t}(t-1)}{t^2}.$$

Exam problem 94

(a) Let $\sqrt{u} = z$. Then $u = z^2$ and $du = 2z\,dz$, and we get

$$\begin{aligned}\int \frac{1}{(z^2-1)z}\,2z\,dz = 2\int \frac{dz}{z^2-1} &= \int \left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz \\ &= \ln|z-1| - \ln|z+1| + C \\ &= \ln\left|\frac{z-1}{z+1}\right| + C = \ln\left|\frac{\sqrt{u}-1}{\sqrt{u}+1}\right| + C.\end{aligned}$$

(b) With $w = \sqrt{e^y+1}$ we get $w^2 = e^y + 1$ and $2w\,dw = e^y\,dy = (w^2-1)\,dy$. Hence,

$$\begin{aligned}\int \frac{1}{\sqrt{e^y+1}}\,dy &= \int \frac{1}{w}\cdot\frac{2w\,dw}{w^2-1} = \int \frac{2\,dw}{w^2-1} = \int \left[\frac{1}{w-1} - \frac{1}{w+1}\right] dw + C \\ &= \ln\left|\frac{w-1}{w+1}\right| + C = \ln\left|\frac{\sqrt{e^y+1}-1}{\sqrt{e^y+1}+1}\right| + C.\end{aligned}$$