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Problem solutions for Seminar 11, 22–27 April

Exam problem 38

(a) Computing differentials, we get

$$\begin{aligned} v\,d(u^2) + u^2\,dv - du &= 3x^2\,dx + 6y^2\,dy \\ e^{ux}\,d(ux) &= y\,dv + v\,dy, \end{aligned}$$

that is,

$$\begin{aligned} 2uv\,du + u^2\,dv - du &= 3x^2\,dx + 6y^2\,dy \\ ue^{ux}\,dx + xe^{ux}\,du &= y\,dv + v\,dy. \end{aligned}$$

If we substitute the values $x = 0$, $y = 1$, $u = 2$, and $v = 1$, we get

$$\begin{aligned} 4\,du + 4\,dv - du &= 6\,dy \\ 2\,dx + 0\,du &= dv + dy. \end{aligned}$$

After a bit of calculation this yields

$$du = -\frac{8}{3}\,dx + \frac{10}{3}\,dy \qquad \text{and} \qquad dv = 2\,dx - dy$$

at the point P. Hence, at this point

$$\frac{\partial u}{\partial y} = \frac{10}{3} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 2.$$

(b) We get

$$\Delta u \approx du = -\frac{8}{3}\,dx + \frac{10}{3}\,dy = -\frac{8}{3} \cdot 0.1 + \frac{10}{3} \cdot (-0.2) = -\frac{2.8}{3} \approx -0.933$$

and

$$\Delta v \approx dv = 2\,dx - dy = 2 \cdot 0.1 - (-0.2) = 0.4.$$

Exam problem 54

The first- and second-order partial derivatives of f are

$$\begin{aligned} &f_1'(x, y) = 2x - y - 3x^2, \qquad f_2'(x, y) = -2y - x, \\ &f_{11}''(x, y) = 2 - 6x, \qquad f_{12}''(x, y) = -1, \qquad f_{22}''(x, y) = -2. \end{aligned}$$

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The stationary points are the solutions of the equation system

$$\begin{aligned} 2x - y - 3x^2 &= 0 \\ -2y - x &= 0 \end{aligned}$$

The last equation is equivalent to $x = -2y$, and if we use this in the first equation we get

$$-5y - 12y^2 = 0 \iff -12y\Big(y + \frac{5}{12}\Big) = 0 \iff y = 0 \text{ eller } y = -\frac{5}{12}\,.$$

It follows that f has two stationary points,

$$(x_1, y_1) = (0, 0) \quad \text{og} \quad (x_2, y_2) = (5/6, -5/12).$$

In order to determine what kind of stationary point they are, we use the second-derivative test and calculate the values of $A = f''_{11}(x, y)$, $B = f''_{12}(x, y)$ and $C = f''_{22}(x, y)$ at each of the three stationary points. That gives the results

(x, y)	A	B	C	$AC - B^2$	Type of stat. point
$(0, 0)$	2	-1	-2	-5	Saddle point
$(\frac{5}{6}, -\frac{5}{12})$	-3	-1	-2	5	Local max. point

Exam problem 141

(a) The derivatives of order one and two are

$$\begin{aligned} &f'_1(x, y) = xe^y - x^2, \qquad f'_2(x, y) = \tfrac{1}{2}x^2e^y - (1 + 3y)e^{3y}, \\ &f''_{11}(x, y) = e^y - 2x, \qquad f''_{12}(x, y) = xe^y, \qquad f''_{22}(x, y) = \tfrac{1}{2}x^2e^y - (6 + 9y)e^{3y}. \end{aligned}$$

(b) The stationary points are the solutions of the equation system

$$xe^y - x^2 = 0 \qquad (1)$$

$$\tfrac{1}{2}x^2e^y - (1 + 3y)e^{3y} = 0 \qquad (2)$$

From (1) we get $x = 0$ or $e^y = x$.

A. If $x = 0$, then (2) gives $y = -1/3$, and we get the stationary point $(x_1, y_1) = (0, -\frac{1}{3})$.

B. If $e^y = x$, then $e^{3y} = x^3$, and (2) yields

$$\tfrac{1}{2}x^3 - (1 + 3y)x^3 = 0 \iff x^3(\tfrac{1}{2} - 1 - 3y) = 0.$$

Since $x = e^y$, we must have $x \neq 0$, and therefore $\frac{1}{2} - 1 - 3y = 0 \iff y = -1/6$. Then $x = e^y = e^{-1/6}$, and we have a second stationary point $(x_2, y_2) = (e^{-1/6}, -\frac{1}{6})$.

We classify the stationary points by means of the second-derivative test.

(x, y)	A	B	C	$AC - B^2$	Type of point
$(0, -\frac{1}{3})$	$e^{-1/3}$	0	$-3e^{-1}$	$-3e^{-4/3}$	Saddle point
$(e^{-1/6}, -\frac{1}{6})$	$-e^{-1/6}$	$e^{-1/3}$	$-4e^{-1/2}$	$3e^{-2/3}$	Local max. point

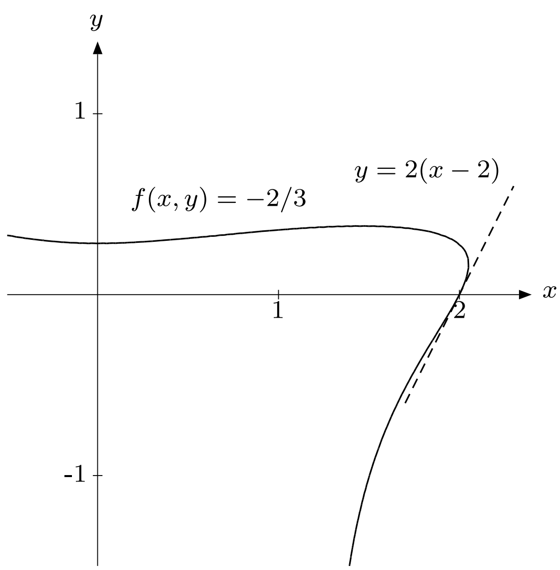
The function f has no global extreme points. Recall that a global extreme point must also be a local extreme point, and the only local extreme point we have here is the local maximum point (x_2, y_2), which gives the local maximum value $f(x_2, y_2) = f(e^{-1/6}, -\frac{1}{6}) = \frac{1}{3}e^{-1/2}$. But this is not a global maximum value for f because, for example, $f(-1, 0) = \frac{5}{6} > \frac{1}{3} > f(x_2, y_2)$.

However, the easiest way to show that f has no global maximum or minimum is to study

$$f(x, 0) = \frac{x^2}{2} - \frac{x^3}{3} = x^3\Big(\frac{1}{2x} - \frac{1}{3}\Big).$$

It is clear from the last expression that

$$\lim_{x\to\infty} f(x, 0) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x, 0) = \infty.$$



Exam problem 141(c)

(c) The slope of the level curve $f(x, y) = -\frac{2}{3}$ at $(2, 0)$ is

$$-\frac{f_1'(2, 0)}{f_2'(2, 0)} = 2.$$

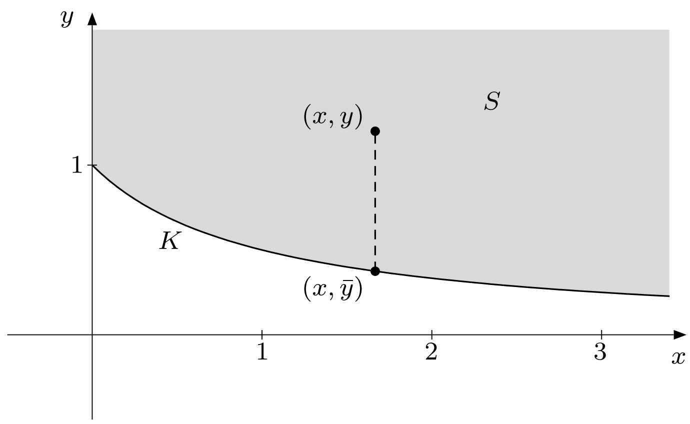
The figure shows the level curve together with its tangent at $(2, 0)$.

Exam problem 24

(a) To simplify the notation, we write $u = -2x - x^2 - 2y^2$. Then the first- and second-order derivatives of $f(x,y) = e^{-2x-x^2-2y^2} = e^u$ are

$$f'_1(x,y) = -2(1+x)e^u, \quad f'_2(x,y) = -4ye^u,$$
$$f''_{11}(x,y) = -2e^u + 4(1+x)^2e^u, \quad f''_{12}(x,y) = -2(1+x)(-4y)e^u,$$
$$f''_{22}(x,y) = -4e^u + (-4y)^2e^u.$$

Since $e^u > 0$ for all u, the only stationary point is $(x_0, y_0) = (-1, 0)$. The corresponding value of u is $u_0 = -2x_0 - x_0^2 - 2y_0^2 = 1$. With $A = f''_{11}(x_0,y_0) = -2e^{u_0} = -2e$, $B = f''_{12}(x_0,y_0) = 0$, and $C = f''_{22}(x_0,y_0) = -4e^{u_0} = -4e$, we have $A < 0$ and $AC - B^2 = 8e^2 > 0$. Therefore $(-1,0)$ is a local maximum point for f.



The set S in Problem 24(b)

(c) The problem "maximize $f(x,y)$ for (x,y) in S" has no solution in the interior of S, since f has no stationary point there. The maximum is therefore attained somewhere on the boundary of S.

It is clear from the expression for $f'_2(x,y)$ that $f(x,y)$ is strictly decreasing with respect to y along each vertical line in S. Thus, for any point (x,y) of S we have $f(x,y) \le f(x,\bar{y})$, where $\bar{y} = 1/(1+x)$. Hence, if $f(x,y)$ has a maximum point over S, then that maximum point must be somewhere on the curve K given by $y = 1/(1+x)$.

Along K we have $f(x,y) = e^{v(x)}$, where $v(x) = -2x - x^2 - 2(1+x)^{-2}$, so we must study $\varphi(x) = e^{v(x)}$ for $x \ge 0$. The derivative of φ is

$$\varphi'(x) = e^{v(x)}v'(x) = e^{v(x)}(-2 - 2x + 4(1+x)^{-3}) = e^{v(x)}\,\frac{4 - 2(1+x)^4}{(1+x)^3}\,.$$

We see that φ has only one stationary point in $[0,\infty)$, namely $x_2 = \sqrt[4]{2} - 1$. Also, $\varphi'(x)$ has the same sign as $2 - (1+x)^4$, so φ is strictly increasing in $[0, x_2]$ and strictly decreasing in $[x_2, \infty)$. Thus, $\varphi(x)$ attains its greatest value for $x = x_2$. The corresponding value of y is $y_2 = 1/(1+x_2) = 1/\sqrt[4]{2}$.

Conclusion: If $f(x, y)$ attains a maximum over S, then the maximum point must be (x_2, y_2) and

$$f_{\max} = f(x_2, y_2) = f\Big(\sqrt[4]{2} - 1, \frac{1}{\sqrt[4]{2}}\Big) = e^{1-2\sqrt{2}} \quad (\approx 0.1607)\,.$$

(d) Why does f have a maximum over S? This was answered almost completely in part (c). It was shown there that for every point (x, y) in S we have

$$f(x, y) \leq f(x, 1/(1 + x)) = \varphi(x).$$

It follows that

$$f(x, y) \leq \varphi(x_2) = f(x_2, y_2)$$

Hence (x_2, y_2) is a maximum point (and the only one) for f over S.

There is no minimum point for f over S. Choosing x and y sufficiently large, we can get $f(x, y)$ as close to 0 as we like, but $f(x, y)$ will always be greater than 0.

Exam problem 110

(a) With the Lagrangian

$$\mathcal{L}(x, y, z) = x^2 + y^2 + z - \lambda(x^2 + 2y^2 + 4z^2 - 1)$$

we get the following necessary first-order conditions:

$$\begin{aligned}
&(1) \qquad & \mathcal{L}_1'(x, y, z) &= 2x - 2\lambda x = 0 \\
&(2) \qquad & \mathcal{L}_2'(x, y, z) &= 2y - 4\lambda y = 0 \\
&(3) \qquad & \mathcal{L}_3'(x, y, z) &= \;\; 1 - 8\lambda z = 0 \\
&(4) \qquad & x^2 + 2y^2 + 4z^2 &= 1
\end{aligned}$$

(Equation (4) is the constraint.) Equation (1) yields $2x(1 - \lambda) = 0$, so there are two cases to investigate:

$$\text{(A)} \quad \underline{x = 0}, \qquad\qquad \text{(B)} \quad \underline{\lambda = 1}.$$

(A) Assume $\underline{x = 0}$. From (2) we get $2y(1 - 2\lambda) = 0$, and thus $\underline{y = 0}$ or $\underline{\lambda = 1/2}$.

(A.1) If $y = 0$, then (4) implies $4z^2 = 1 - x^2 - 2y^2 = 1$. Therefore $z^2 = 1/4$, and $z = \pm 1/2$. Equation (3) gives $\lambda = 1/8z$. We get the following candidates for extreme points:

$$\begin{aligned}
&P_1 : (0, 0, 1/2) \text{ with } \lambda = 1/4, \qquad && f(0, 0, 1/2) = 1/2, \\
&P_2 : (0, 0, -1/2) \text{ with } \lambda = -1/4, \qquad && f(0, 0, -1/2) = -1/2.
\end{aligned}$$

(A.2) If $\lambda = 1/2$, then (3) yields $z = 1/8\lambda = 1/4$. It then follows from (4) that $2y^2 = 1 - x^2 - 4z^2 = 1 - 0 - 1/4 = 3/4$ (remember that we have assumed $x = 0$!), and hence $y = \pm\sqrt{3/8} = \pm\sqrt{6}/4$. This gives the candidate points

$$\begin{aligned}
&P_3 : (0, \sqrt{6}/4, 1/4) \text{ med } \lambda = 1/2, \qquad && f(0, \sqrt{6}/4, 1/4) = 5/8, \\
&P_4 : (0, -\sqrt{6}/4, 1/4) \text{ med } \lambda = 1/2, \qquad && f(0, -\sqrt{6}/4, 1/4) = 5/8.
\end{aligned}$$

(B) Now assume $\underline{\lambda = 1}$. Equation (3) gives $z = 1/8$, and (2) gives $y = 0$. From the constraint (4) we get $x^2 = 1 - 2y^2 - 4z^2 = 1 - 4/64 = 15/16$, and therefore $x = \pm\sqrt{15}/4$. Candidate points:

$$P_5 : (\sqrt{15}/4, 0, 1/8) \text{ with } \lambda = 1, \qquad f(\sqrt{15}/4, 0, 1/8) = 17/16,$$
$$P_6 : (-\sqrt{15}/4, 0, 1/8) \text{ with } \lambda = 1, \qquad f(-\sqrt{15}/4, 0, 1/8) = 17/16.$$

Comparing the function values, we see that f attains its maximum value $f_{\text{maks}} = 17/16$ at the points P_5 and P_6, and its minimum value $f_{\min} = -1/2$ at P_2. (The admissible set, i.e. the set of points that satisfy the constraint, is closed and bounded, and since f is continuous, the extreme value theorem guarantees that f will attain both a maximum and a minimum over the admissible set.)

(b) The change in the maximum value is

$$\Delta f^* = f^*(1 + 0.02) - f^*(1) \approx \lambda\, dc = 1 \cdot 0.02 = 0.02,$$

cf. formula (14.2.3) on page 496 in EMEA (formula (14.2.5) on page 505 in MA I).

Solutions of the extra problems:

Exam problem 16

(a) $f_1'(x,y) = 2(x + y - 2) + 2(x^2 + y - 2)2x = 4x^3 + 4xy - 6x + 2y - 4,$
$f_2'(x,y) = 2(x + y - 2) + 2(x^2 + y - 2) = 2x^2 + 2x + 4y - 8,$
$f_{11}''(x,y) = 12x^2 + 4y - 6, \quad f_{12}''(x,y) = f_{21}''(x,y) = 4x + 2, \quad f_{22}''(x,y) = 4.$

(b) The stationary points are where $f_1'(x,y) = 0$ and $f_2'(x,y) = 0$. From the latter equation we get $2y = -x^2 - x + 4$, and if we substitute this expression for $2y$ in the first equation we get

$$4x^3 + 2x(-x^2 - x + 4) - 6x + (-x^2 - x + 4) - 4 = 0 \iff 2x^3 - 3x^2 + x = 0$$
$$\iff x = 0 \text{ or } 2x^2 - 3x + 1 = 0 \iff x = 0 \text{ or } x = 1 \text{ or } x = 1/2.$$

It follows that the stationary points of f are

$$(x_1, y_1) = (0, 2), \quad (x_2, y_2) = (1, 1), \quad (x_3, y_3) = (1/2, 13/8).$$

The second-derivative test yields the following results:

(x,y)	$A = f_{11}''$	$B = f_{12}''$	$C = f_{22}''$	$AC - B^2$	Type of point
$(0,2)$	2	2	4	4	Local min. point
$(1,1)$	10	6	4	4	Local min. point
$(1/2, 13/8)$	7/2	4	4	-2	Saddle point

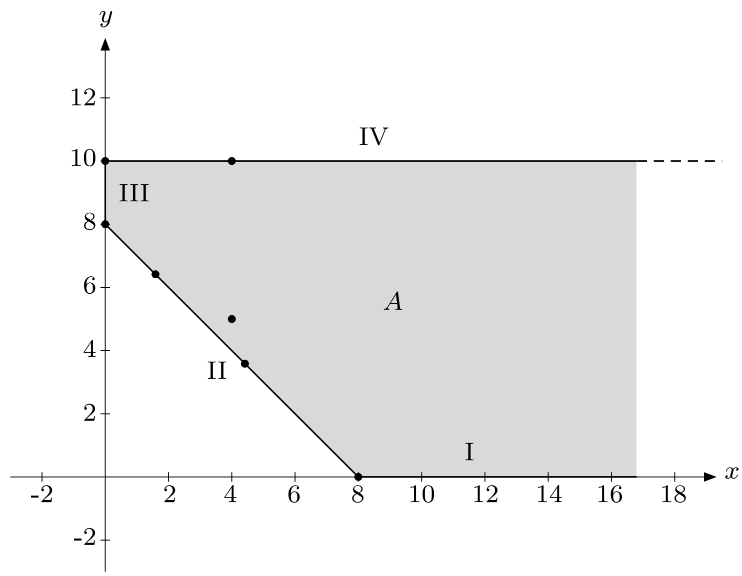
This shows that the points (x_1, y_1) and (x_2, y_2) are local minimum points. They are, in fact, global minimum points because $f(x_1, y_1) = f(x_2, y_2) = -8$ and it is clear from the definition of f that $f(x, y) \geq 0 + 0 - 8 = -8$ for all (x, y).

(c) $g'(t) = pf_1'(pt, qt) + qf_2'(pt, qt) = 4p^4t^3 + 6p^2qt^2 + (-6p^2 + 4q^2 + 4pq)t - 4p - 8q$.

If $p \neq 0$, then the t^3 term will dominate for large t and $g'(t) \to \infty$ as $t \to \infty$.

If $p = 0$, then $q \neq 0$ and $g'(t) = 4q^2t - 8q$ will obviously tend to ∞ as $t \to \infty$.

Exam problem 33



Exam problem 33 (a)

(b) Since f has partial derivatives everywhere, any minimum points in the set A must be stationary points in A or boundary points of A. The first-order partial derivatives of f are

$$\frac{\partial f}{\partial x} = 2x - 8, \qquad \frac{\partial f}{\partial y} = -y^2 + 8y - 15.$$

It is clear that $f_x' = 0$ if and only if $x = 4$, and $f_y' = 0$ if and only if $y = 3$ or $y = 5$. Thus, the stationary points of f are $(4, 3)$ and $\underline{(4, 5)}$. Only the last of these belongs to A.

It is natural to see the boundary of A as composed of four parts, namely the edges labeled I, II, III, and IV in the figure. We shall investigate these parts separately.

Along I, $y = 0$, $x \geq 8$, and $f(x, y) = f(x, 0) = x^2 - 8x$. Let $g(x) = x^2 - 8x$. Then $g'(x) = 2x - 8$. Since $g'(x) > 0$ for all $x > 8$, the value of $f(x, y)$ will increase as we move right along the edge I. This means that the minimum point of f along I is $\underline{(8, 0)}$.

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Along II, $y = 8 - x$ and $0 \le x \le 8$. Let

$$h(x) = f(x, 8-x) = -\frac{1}{3}(8-x)^3 + 4(8-x)^2 - 15(8-x) + x^2 - 8x$$
$$= \cdots = \frac{x^3}{3} - 3x^2 + 7x - \frac{104}{3}\,.$$

A minimum point for $f(x, y)$ along II must correspond to a minimum point for $h(x)$ over the interval $[0, 8]$. Since

$$h'(x) = x^2 - 6x + 7 = (x-3)^2 - 2,$$

thee stationary points of h are $x = 3 \pm \sqrt{2}$. A minimum point for h over $[0, 8]$ must be one of these points or an end point of the interval. Calculation of function values gives

$$\begin{aligned} h(0) &= -104/3 \approx -34.6667, \\ h(3-\sqrt{2}) &= -(95 - 4\sqrt{2})/3 \approx -29.7810, \\ h(3+\sqrt{2}) &= -(95 + 4\sqrt{2})/3 \approx -33.5523, \\ h(8) &= 0. \end{aligned}$$

Hence, the minimum point of f along II is $\underline{(0, 8)}$.

Along III we have $x = 0$ and $8 \le y \le 10$. Here $f(x, y) = f(0, y) = -y^3/3 + 4y^2 - 15y$. Since

$$\frac{\partial}{\partial y}\left(-\frac{y^3}{3} + 4y^2 - 15y\right) = -y^2 + 8y - 15 = -(y-3)(y-5) < 0$$

when $y > 5$, the value of $f(x, y)$ will decrease when we move upwards along III, and so the minimum point of f along III is $\underline{(0, 10)}$.

Along IV, $y = 10$ and $x \ge 0$. Here

$$f(x, y) = f(x, 10) = -\frac{250}{3} + x^2 - 8x,$$

which attains its lowest value when $\underline{x = 4}$ (and $\underline{y = 10}$).

The minimum point of f over A must be among the five points that we have found. Since

$$f(4,5) = -98/3, \quad f(8,0) = 0, \quad f(0,8) = -104/3,$$
$$f(0,10) = -250/3, \quad \text{and} \quad f(4,10) = -298/3,$$

it is clear that the minimum point of f over A is $\underline{(4, 10)}$, and the minimum value is $\underline{-298/3}$.

The figure shows these five points together with the two points $(3 \pm \sqrt{2}, 5 \mp \sqrt{2})$ on II, which correspond to the stationary points of the function h.

Exam problem 61

We are going to investigate the equation system

$$\begin{aligned} \ln(x+u) + uv - y^2 e^v + y &= 0 \\ u^2 - x^v &= v \end{aligned}$$

around the point $P : (x, y, u, v) = (2, 1, -1, 0)$.

(a) When we differentiate the system, we shall need the differential of x^v. The simplest way to find this is to use that $x^v = (e^{\ln x})^v = e^{v \ln x}$. We get

$$d(x^v) = d(e^{v\ln x}) = e^{v\ln x} d(v \ln x) = x^v\big(\ln x\, dv + v\, d(\ln x)\big) = x^v\big(\ln x\, dv + \frac{v}{x}\, dx\big),$$

and if we now differentiate the given equation system we get

$$\begin{aligned} \frac{1}{x+u}(dx+du) + v\,du + u\,dv - 2ye^v\,dy - y^2e^v\,dv + dy &= 0 \\ 2u\,du - x^v \ln x\,dv - x^{v-1}v\,dx &= dv \end{aligned}$$

(b) Since we only want the values of the partial derivatives at the point P, we insert the values of x, y, u, and v in the differentiated equation system. Thus we shall not bother with finding general expressions for du og dv, but only their values at P. With $x = 2$, $y = 1$, $u = -1$, and $v = 0$ we get

$$\begin{aligned} \frac{1}{1}(dx+du) - dv - 2y\,dy - dv + dy &= 0 \\ -2\,du - \ln 2\,dv - 0 &= dv \end{aligned}$$

We rearrange this as

$$\begin{aligned} du - 2\,dv &= -dx + dy \\ -2u - (1+\ln 2)\,dv &= 0 \end{aligned}$$

The last equation yields

$$(*) \qquad dv = -\frac{2}{1+\ln 2}\,du,$$

and if we insert this into the first equation, we get

$$du + \frac{4}{1+\ln 2}\,du = -dx + dy \iff \frac{5+\ln 2}{1+\ln 2}\,du = -dx + dy.$$

Hence,

$$du = -\frac{1+\ln 2}{5+\ln 2}\,dx + \frac{1+\ln 2}{5+\ln 2}\,dy,$$

and therefore

$$u'_x = -\frac{1+\ln 2}{5+\ln 2}\,, \qquad u'_y = \frac{1+\ln 2}{5+\ln 2}\,.$$

Equation (∗) gives

$$dv = \frac{2}{5+\ln 2}\,dx - \frac{2}{5+\ln 2}\,dy,$$

and so

$$v'_x = \frac{2}{5+\ln 2}\,, \qquad v'_y = -\frac{2}{5+\ln 2}\,.$$

(c) We use the "increment formula" (Norwegian: tilvekstformelen):

$$u(x+dx, y+dy) \approx u(x,y) + u'_x(x,y)\,dx + u'_y(x,y)\,dy.$$

This formula yields

$$\begin{aligned} u(2-0.01, 1+0.02) &\approx u(2,1) + u'_x(2,1)\cdot(-0.01) + u'_y(2,1)\cdot 0.02 \\ &= -1 + \Big(-\frac{1+\ln 2}{5+\ln 2}\Big)(-0.01) + \frac{1+\ln 2}{5+\ln 2}\cdot 0.02 \\ &= -1 + \frac{1+\ln 2}{5+\ln 2}\cdot 0.03 \approx -0.9911. \end{aligned}$$

Exam problem 67

(a) Vi bruker implisitt elastisitering. Av $\mathrm{El}_x(y^2 + e^{x+1/y}) = \mathrm{El}_x\, 3 = 0$ får vi

$$\mathrm{El}_x\, y^2 + \mathrm{El}_x\, e^x + \mathrm{El}_x\, e^{1/y} = 0,$$

som gir

$$\begin{aligned} 2\,\mathrm{El}_x\, y + x + \frac{1}{y}\,\mathrm{El}_x\left(\frac{1}{y}\right) = 0 \;&\overset{(*)}{\iff}\; \left(2 - \frac{1}{y}\right)\mathrm{El}_x\, y = -x \\ &\iff\; \mathrm{El}_x\, y = \frac{x}{\frac{1}{y}-2} = \frac{xy}{1-2y}\,. \end{aligned}$$

(Ved (∗) bruker vi kjerneregelen, som gir $\mathrm{El}_x\, e^u = \mathrm{El}_u\, e^u\, \mathrm{El}_x\, u = u\,\mathrm{El}_x\, u$, med $u = 1/y$.)

(b) Differensiering gir likningssystemet

$$\begin{aligned} \alpha u^{\alpha-1}\,du + \beta v^{\beta-1}\,dv &= 2^\beta\,dx + 3y^2\,dy \\ \alpha u^{\alpha-1}v^\beta\,du + u^\alpha \beta v^{\beta-1}\,dv - \beta v^{\beta-1}\,dv &= dx - dy \end{aligned}$$

I punktet $P = (x,y,u,v) = (1,1,1,2)$ får vi

$$\begin{aligned} \alpha\,du + \beta 2^{\beta-1}\,dv &= 2^\beta\,dx + 3\,dy \\ \alpha 2^\beta\,du \qquad\qquad &= dx - dy \end{aligned}$$

som gir

$$\begin{aligned} du &= \frac{2^{-\beta}}{\alpha}\,dx - \frac{2^{-\beta}}{\alpha}\,dy \\ dv &= \frac{2^\beta - 2^{-\beta}}{\beta 2^{\beta-1}}\,dx + \frac{3+2^{-\beta}}{\beta 2^{\beta-1}}\,dy \end{aligned}$$

Dermed har vi

$$\frac{\partial u}{\partial x} = \frac{2^{-\beta}}{\alpha}\,, \quad \frac{\partial u}{\partial y} = -\frac{2^{-\beta}}{\alpha}\,, \quad \frac{\partial v}{\partial x} = \frac{2^{\beta} - 2^{-\beta}}{\beta 2^{\beta-1}}\,, \quad \frac{\partial v}{\partial y} = \frac{3 + 2^{-\beta}}{\beta 2^{\beta-1}}\,.$$

(c) Av det foregående ser vi at $u'_x(1,1) = 2^{-\beta}/\alpha$ og $u'_y(1,1) = -2^{-\beta}/\alpha$. Videre har vi $u(1,1) = 1$. Tilvekstformelen gir da

$$\begin{aligned} u(0.99, 1.01) &\approx u(1,1) + u'_x(1,1)\cdot(-0.01) + u'_y(1,1)\cdot 0.01 \\ &= 1 + \frac{2^{-\beta}}{\alpha}\cdot\frac{-1}{100} - \frac{2^{-\beta}}{\alpha}\cdot\frac{1}{100} = 1 - \frac{2^{1-\beta}}{100\alpha}\,. \end{aligned}$$

Exam problem 138

(a) $\mathcal{L} = e^x + y + z - \lambda_1(x + y + z - 1) - \lambda_2(x^2 + y^2 + z^2 - 1)$

$$\frac{\partial \mathcal{L}}{\partial x} = e^x - \lambda_1 - 2\lambda_2 x = 0 \qquad \text{(i)}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 1 - \lambda_1 - 2\lambda_2 y = 0 \qquad \text{(ii)}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 1 - \lambda_1 - 2\lambda_2 z = 0 \qquad \text{(iii)}$$

Fra (ii) og (iii) følger det at $2\lambda_2 y = 2\lambda_2 z$. Dermed er (A) $y = z$ eller (B) $\lambda_2 = 0$.
A Hvis $z = y$, gir bibetingelsene at $x^2 + 2y^2 = 1$ og $x + 2y = 1$. Av den siste likningen finner vi $x = 1 - 2y$ som innsatt i $x^2 + 2y^2 = 1$ og ordnet gir $6y^2 - 4y = 0$. Herav $y = 0$ eller $y = 2/3$. Dette gir kandidatene $(x, y, z) = (1, 0, 0)$ med $\lambda_1 = 1$ og $\lambda_2 = \frac{1}{2}(e - 1)$, og $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ med $\lambda_1 = \frac{1}{3} + \frac{2}{3}e^{-1/3}$ og $\lambda_2 = \frac{1}{2} - \frac{1}{2}e^{-1/3}$.

B Hvis $\lambda_2 = 0$, gir (ii) at $\lambda_1 = 1$ som innsatt i (i) gir $e^x = 1$, og dermed $x = 0$. Bibetingelsene gir da $y^2 + z^2 = 1$ og $y + z = 1$ med løsninger $(y, z) = (0, 1)$ og $(1, 0)$. Det gir kandidatene $(x, y, z) = (0, 0, 1)$, $(0, 1, 0)$ med tilhørende $\lambda_1 = 1$, $\lambda_2 = 0$.
For $(1, 0, 0)$ er kriteriefunksjonen lik e.
For $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ er kriteriefunksjonen lik $e^{-1/3} + \frac{4}{3}$.
For $(0, 0, 1)$ er kriteriefunksjonen lik 2.
For $(0, 1, 1)$ er kriteriefunksjonen lik 2. Her er $e^{-1/3} + \frac{4}{3} < 1 + \frac{4}{3} = \frac{7}{3} < e$. Siden beskrankningsmengden er lukket og begrenset og kriteriefunksjonen er kontinuerlig, fins det et maksimum, og det er i punktet $(1, 0, 0)$.

(b) $\Delta f^* \approx \lambda_1 \cdot (0.02) + \lambda_2 \cdot (-0.02) = 0.02 - 0.002 \cdot \dfrac{1}{2}(e - 1) = 0.01(3 - e)$.