Department of Economics
Knut Sydsæter, A. Strøm

Answers to the examination problems in ECON3120/4120, 4 December 2003

Problem 1

(a) $f'(x) = ae^{-bx} + (ax+1)(-b)e^{-bx} = abe^{-bx}\big(\frac{1}{b} - \frac{1}{a} - x\big)$,

$f''(x) = -ab^2e^{-bx}\big(\frac{1}{b} - \frac{1}{a} - x\big) - abe^{-bx} = ab^2e^{-bx}\big[x - \big(\frac{2}{b} - \frac{1}{a}\big)\big]$

(b) We see that $f'(x) = 0$ when $x = x^* = \frac{1}{b} - \frac{1}{a}$. Moreover, $f'(x) > 0$ if $x < x^*$, $f'(x) < 0$ if $x > x^*$. It follows that $f(x)$ is increasing in $(-\infty, x^*]$ and decreasing in $[x^*, \infty)$. Hence, x^* is a (global) maximum point for f.

$f''(x) = 0$ when $x = x^{**} = \frac{2}{b} - \frac{1}{a}$, and $f''(x)$ changes sign around x^{**}. Thus x^{**} is an inflection point. Also, $x^{**} = \frac{1}{b} - \frac{1}{a} + \frac{1}{b} = x^* + \frac{1}{b}$.

(c) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{ax+1}{e^{bx}} = \text{“}\frac{\infty}{\infty}\text{”} = \lim_{x\to\infty} \frac{a}{be^{bx}} = 0$, by l'Hôpital's rule.

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (ax+1)\,e^{-bx} = -\infty$. (The first factor tends to $-\infty$, the second to ∞.)

(d)
$$\begin{aligned}
\int f(x)\,dx &= \int (ax+1)e^{-bx}\,dx \\
&= -\frac{1}{b}(ax+1)e^{-bx} - \int \big(-\frac{1}{b}\big)\,a\,e^{-bx}\,dx \\
&= -\frac{1}{b}(ax+1)e^{-bx} - \frac{a}{b^2}e^{-bx} + C.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^\infty f(x)\,dx &= \lim_{A\to\infty} \int_0^A f(x)\,dx \\
&= \lim_{A\to\infty} \Big|_0^A \Big[\frac{-(ax+1)}{b}\,e^{-bx} - \frac{a}{b^2}\,e^{-bx}\Big] \\
&= \lim_{A\to\infty} \Big[\frac{-(aA+1)e^{-bA}}{b} - \frac{a}{b^2}e^{-bA} + \frac{1}{b} + \frac{a}{b^2}\Big] \\
&= \frac{a+b}{b^2}\,.
\end{aligned}$$

m2h03f 6.5.2004 833

Problem 2

(a) Define $F(x,y,z) = ze^z - xy$. Then

$$
\begin{aligned}
z_1'(x,y) &= -\frac{F_1'(x,y,z)}{F_3'(x,y,z)} = -\frac{-y}{e^z + ze^z} = \frac{y}{e^+ ze^z} = \frac{e}{e+e} = \frac{1}{2}\,, \\
z_2'(x,y) &= -\frac{F_2'(x,y,z)}{F_3'(x,y,z)} = -\frac{-x}{e^z + ze^z} = \frac{x}{e^+ ze^z} = \frac{1}{2e}\,, \\
z_{12}''(x,y) &= \frac{\partial}{\partial y} z_1'(x,y) = \frac{\partial}{\partial y}\frac{y}{e^z + ze^z} \\
&= \frac{(e^z + ze^z) - y(e^z\, z_2' + z_2'\, e^z + ze^z\, z_2')}{(e^+ ze^z)^2} \\
&= \frac{(e+e) - e\big(e\,\frac{1}{2e} + \frac{1}{2e}e + e\,\frac{1}{2e}\big)}{(2e)^2} = \frac{1}{8e}\,,
\end{aligned}
$$

Alternative method: Taking logarithms, we see that $ze^z = xy$ implies

$$\ln z + z = \ln x + \ln y.$$

Differentiation w.r.t. x yields

$$(*) \qquad \frac{1}{z} z_1' + z_1' = \frac{1}{x} \iff z_1' + zz_1' = \frac{z}{x} \iff z_1' = \frac{z}{x(1+z)}\,,$$

so $z_1'(1,e) = 1/2$. Symmetrically,

$$z_2'(1,e) = \frac{z}{y(1+z)} = \frac{1}{2e}\,.$$

In $(*)$, we found $z_1' + zz_1' = \frac{1}{x}z$. Differentiation w.r.t. y gives

$$z_{12}'' + z_2' z_1' + zz_{12}'' = \frac{1}{x} z_2'.$$

If we insert $x = 1$, $z = 1$, $z_1' = \frac{1}{2}$, and $z_2' = \frac{1}{2e}$, we get $z_{12}'' = \frac{1}{8e}$.

Problem 3

Let $z = 1 - w$. Then $-dw = dz$, so that

$$
\begin{aligned}
I &= \int \frac{w}{(1-w)^3}\,dw = \int \frac{1-z}{z^3}\,(-dz) = \int \Big(\frac{1}{z^2} - \frac{1}{z^3}\Big)\,dz \\
&= \int (z^{-2} - z^{-3})\,dz = -\frac{1}{z} + \frac{1}{2z^2} + C = \frac{1-2z}{2z^2} + C \\
&= \frac{1 - 2(1-w)}{2(1-w)^2} + C = \frac{2w-1}{2(1-w)^2} + C
\end{aligned}
$$

Problem 4

(a) $|\mathbf{A}| = 11 \cdot (-10) - 18(-6) = -110 + 108 = -2$.

$$\mathbf{A}^2 = \begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix} \begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix} = \begin{pmatrix} 13 & -6 \\ 18 & -8 \end{pmatrix},$$

so

$$\mathbf{A}^2 - 2\mathbf{I}_2 = \begin{pmatrix} 13 & -6 \\ 18 & -8 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix} = \mathbf{A}.$$

Hence, $\mathbf{A}^2 + c\mathbf{A} = 2\mathbf{I}_2$ if $c = -1$.

(b) If $\mathbf{B}^2 = \mathbf{A}$, then $|\mathbf{B}^2| = |\mathbf{A}| = -2$. But $|\mathbf{B}^2| = |\mathbf{B}| \cdot |\mathbf{B}| = |\mathbf{B}|^2$, which cannot be negative. Contradiction.

Problem 5

The Lagrangian for this problem is

$$\mathcal{L}(x, y) = \ln(2 + x^2) + y^2 - \lambda(x^2 + 2y - 2).$$

Hence, the necessary first-order conditions for (x, y) to be a minimum point are

$$(1)\ \frac{\partial \mathcal{L}}{\partial x} = \frac{2x}{2 + x^2} - 2\lambda x = 0, \quad (2)\ \frac{\partial \mathcal{L}}{\partial y} = 2y - 2\lambda = 0, \quad (3)\ x^2 + 2y = 2.$$

From (1) we get $x\big(\frac{1}{2 + x^2} - \lambda\big) = 0$, so $x = 0$ or $\lambda = \frac{1}{2 + x^2}$.

A. If $x = 0$, then (3) gives $y = 1$, so $(x_1, y_1) = (0, 1)$ is a candidate.

B. If $x \neq 0$, then $y = \lambda = \frac{1}{2 + x^2}$, where we used (2). Inserting $y = \frac{1}{2 + x^2}$ into (3) gives

$$x^2 + \frac{2}{2 + x^2} = 2 \iff 2x^2 + x^4 + 2 = 4 + 2x^2 \iff x^4 = 2 \iff x = \pm\sqrt[4]{2}.$$

From (3), $y = 1 - \frac{1}{2}x^2 = 1 - \frac{1}{2}\sqrt{2}$. Thus,

$$(*) \qquad (x_2, y_2) = (\sqrt[4]{2}, 1 - \tfrac{1}{2}\sqrt{2}) \quad \text{and} \quad (x_3, y_3) = (-\sqrt[4]{2}, 1 - \tfrac{1}{2}\sqrt{2})$$

are candidates.

Now let $f(x, y) = \ln(2 + x^2) + y^2$. Then $f(x_1, y_1) = f(0, 1) = \ln 2 + 1 \approx 1.69$, while for the candidate points in $(*)$, we get

$$f(x_2, y_2) = f(x_3, y_3) = \ln(2 + \sqrt{2}) + (1 - \tfrac{1}{2}\sqrt{2})^2 = \ln(2 + \sqrt{2}) + \tfrac{3}{2} - \sqrt{2} \approx 1.31.$$

Hence, the minimum points for $f(x, y)$ (subject to $x^2 + 2y = 2$) are (x_2, y_2) and (x_3, y_3).