ECON3120/ECON4120 Mathematics 2 – note on the solution to the fall 2007 exam

Problem 1 Cofactor expansion along the last row gives

$$\begin{aligned}|\mathbf{A}_u| &= -u\begin{vmatrix}1 & 1-u\\ u-1 & 3u-1\end{vmatrix} + 2u\begin{vmatrix}1 & 2u-1\\ u-1 & 1\end{vmatrix}\\ &= u\left(1-3u-u^2+2u-1+2-2(2u-1)(u-1)\right)\\ &= u\left(-u^2-u-4u^2+6u\right)\\ &= 5u^2(1-u).\end{aligned}$$

So – regardless of k – there is a unique solution whenever $u \notin \{0,1\}$. The cases $u=0$ and $u=1$ have to be treated separately.

The case $u=0$ gives

$$\begin{aligned} x - \ y + \ z &= 0\\ -x + \ y - \ z &= k\\ 0x + 0y + 0z &= 0\end{aligned}$$

or – adding the first line to the second (or merely comparing those two!)

$$\begin{aligned} x - \ y + \ z &= 0\\ 0 &= k\end{aligned}$$

So for $u=0$ there is *no solution for $k \neq 0$,* while for $k=0$ there are *two degrees of freedom.*

The case $u=1$ gives

$$\begin{aligned} x + y + \ z &= 1\\ y - 2z &= k\\ y - 2z &= k\end{aligned}$$

where the last two lines are the same, so one of them can be dropped. It is easy to see that – regardless of k – we have *one degree of freedom*; if we choose $z=t$, the equation system becomes

$$\begin{aligned} x + y &= 1-2t\\ y &= k+2t\end{aligned}$$

which – for each t – has a unique solution for x and y. To summarize:

- No solution when $u = 0 \neq k$
- Two degrees of freedom when $u = k = 0$
- One degree of freedom when $u = 1$
- Unique solution otherwise.

Note: In the lectures it was stressed, repeatedly, that a «zero line» does *not* imply infinitely many solutions, and that they have to check the rest of the equation system for consistency – which is also necessary in order to find the number of degrees of freedom. So there is hardly any excuse for not examining the case $u = 0$ in sufficient detail.

Problem 2 We are given (for $t > 0$, $x > 0$):

$$V(t, x) = g(t)h(x)e^{-rt} - x$$

(a) The first-order conditions are

$$\begin{aligned} 0 &= V_t'(t^*, x^*) = h(x^*) \cdot \left(g'(t^*)e^{-rt^*} - rg(t^*)e^{-rt^*}\right) \\ 0 &= V_x'(t^*, x^*) = g(t^*)e^{-rt^*}h'(x^*) - 1 \end{aligned}$$

or:

$$\underline{\underline{g'(t^*) = rg(t^*)}}$$

$$\underline{\underline{h'(x^*) = e^{rt^*}/g(t^*)}}$$

(b) We are given that $V_{tx}''(t^*, x^*) = 0$, so the second-order test will be satisfied if $V_{tt}''(t^*, x^*) < 0$ and $V_{xx}''(t^*, x^*) < 0$.
Consider first the latter: $V_{xx}''(t, x) = g(t)e^{-rt}h''(x)$, and since $g(t) > 0$, this has the same sign as h''. So if $h''(x^*) < 0$, then $V_{xx}''(t^*, x^*) < 0$.
For V_{tt}'', we have

$$\begin{aligned} V_{tt}''(t, x) &= h(x) \cdot \tfrac{d}{dt}\left[(g'(t) - rg(t))e^{-rt}\right] \\ &= h(x) \cdot \left[(g''(t) - rg'(t))e^{-rt} - r(g'(t) - rg(t))e^{-rt}\right] \\ &= h(x) \cdot \left[g''(t) - rg'(t) - r(g'(t) - rg(t))\right] e^{-rt} \end{aligned}$$

which – since $he^{-rt} > 0$ – has the same sign as the expression in the brackets. Now the proposition we are asked to show does not involve g', so we use the hint and substitute $g'(t^*) = rg(t^*)$ from the first-order condition. Then at the stationary point we get that $0 > V_{tt}''(t^*, x^*)$ if

$$\begin{aligned} 0 &> g''(t^*) - rg'(t^*) - r(g'(t^*) - rg(t^*)) \\ &= g''(t^*) - r \cdot rg(t^*) - 0 \end{aligned}$$

so that the second-order condition will hold if

$$r^2 g(t^*) > g''(t^*)$$

which is precisely what we were asked to show.

(c) We are given $g(t) = e^{\sqrt{t}}$ and $h(x) = \ln(x+1)$, which are both positive functions. We have $g'(t) = g(t)/2\sqrt{t}$ and $h'(x) = 1/(x+1)$, so the first-order conditions become

$$\begin{aligned}
\text{For } t^* : \qquad \frac{1}{2\sqrt{t}} g(t^*) &= r g(t^*) \\
\text{implying} \quad t^* &= 1/4r^2 \\
\text{For } x^* : \qquad \frac{1}{x^*+1} &= e^{rt^*} e^{-\sqrt{t}^*} \\
&= e^{\frac{1}{4r} - \frac{1}{2r}} \\
&= e^{-\frac{1}{4r}} \\
\text{implying} \quad x^* &= e^{1/4r} - 1
\end{aligned}$$

For the second-order conditions, we easily see that $h''(x) = -(x+1)^{-2}$ which is < 0 for any x (hence also for x^*), so we only need to verify $r^2 g(t^*) > g''(t^*)$. Now the second derivative is

$$\begin{aligned}
g''(t) &= \tfrac{1}{2} g'(t) t^{-1/2} + (-\tfrac{1}{4}) g(t) t^{-3/2} \\
&= \tfrac{1}{4} \left(2g(t) t^{-1/2} \cdot \tfrac{1}{2} t^{-1/2} - g(t) t^{-3/2} \right) \\
&= \tfrac{1}{4} g(t) t^{-3/2} (t^{1/2} - 1)
\end{aligned}$$

so that the statement $r^2 g(t^*) > g''(t^*)$ is equivalent to

$$\begin{aligned}
r^2 g(t^*) &> \tfrac{1}{4} g(t^*) \cdot (t^*)^{-3/2} \cdot ((t^*)^{1/2} - 1) \\
&\Updownarrow \quad \text{(because } g > 0\text{)} \\
r^2 &> \tfrac{1}{4} (4r^2)^{3/2} \cdot (\frac{1}{2r} - 1)
\end{aligned}$$

The right hand side is equal to $2r^3(\frac{1}{2r} - 1) = r^2 - 2r^3$, which is $< r^2$ when $2r^3 > 0$, and we are given that $r > 0$. So the second-order test implies local maximum, for all $r > 0$.

Problem 3 We are given

$$\dot{x} = (t - K) \frac{x}{\ln x} \qquad (\text{for } t > 0,\ x > 1)$$

(a) A stationary point is where $\dot{x} = 0$, and we see that the right hand side is zero when $\underline{\underline{t = K}}$.

(b) The equation is separable:

$$\frac{\ln x}{x}\,dx = (t - K)\,dt.$$

The most common would probably be to go by way of the indefinite integral first:

$$\int \frac{\ln x}{x}\,dx = \int (t - K)\,dt.$$

For the left hand side integral, the substitution $u = \ln x$ (with $du = dx/x$) gives $\frac{1}{2}(\ln x)^2$ plus a constant; the right hand side is equal to $\frac{1}{2}((t-K)^2 + C)$ (possibly with a different constant). So

$$\begin{aligned}(\ln x(t))^2 &= C + (t-K)^2 \qquad \text{or}\\ \ln x(t) &= \pm\sqrt{C + (t-K)^2}\end{aligned}$$

Insert $t = K$ and $x = e$; then we see that «±» has to be «+», and $C = 1$. So the solution is

$$\underline{\underline{x(t) = e^{\sqrt{1+(t-K)^2}}}}$$

Note: There might be a few students who would take the positive square root without giving any comment why there is a negative one to be eliminated. It has been announced that such issues play a minor rôle in this course, and it should therefore only lead to a very small reduction in score.

Problem 4 We are given the problem (for strictly positive A, B and C)

$$\max_{(x,y)} 4e^x + \tfrac{1}{2}Ax^2y^2 + e^{3y} \qquad \text{subject to} \qquad x^2 + By^2 \le C, \qquad x \ge 0 \qquad \text{and} \quad y \ge 0.$$

(a) The Lagrangian is:

$$L(x,y) = 4e^x + \tfrac{1}{2}Ax^2y^2 + e^{3y} - \lambda(x^2 + By^2) + \mu x + \nu y.$$

The Kuhn-Tucker conditions are

$$\boxed{\begin{array}{ll} 0 = 4e^x + Axy^2 - 2\lambda x^2 + \mu & (1)\\ 0 = 3e^{3y} + Ax^2y - 2\lambda By^2 + \nu & (2)\\ \lambda \ge 0 \quad \text{with } \lambda = 0 \text{ if } x^2 + By^2 < C & (3)\\ \mu \ge 0 \quad \text{with } \mu = 0 \text{ if } x > 0 & (4)\\ \nu \ge 0 \quad \text{with } \nu = 0 \text{ if } y > 0 & (5)\end{array}}$$

Note: The alternative conditions involving complimentary slackness in the first order conditions (i.e. merging (4) into (1) and (5) into (2)) are only briefly mentioned, and the students are not in any way expected to use them, though it is of course allowed. Also, some students will write e.g. (5) as «$\nu \geq 0 \quad (= 0 \text{ if } y > 0) \quad (\text{and } y \geq 0 \text{ always})$», which, although not the usual Kuhn-Tucker form, is fully acceptable (and was even recommended for students who prefer to do so to keep track of the substantial conditions.)

(b) To show that $x^2 + By^2 = C$ and $xy \neq 0$, we shall show that we get a contradiction if not:

- If $x = 0$ then (1) becomes $0 = 4e^x + \mu$, which is impossible since $e^x > 0$ and $\mu \geq 0$.
- If $y = 0$ then (2) becomes $0 = 3e^{3y} + \nu$, impossible by a similar argument.
- If $x^2 + By^2 \neq C$ (that is, $< C$), then $\lambda = 0$ by (3). In the same manner as above, the right hand side of (1) will be the sum of nonnegative terms $4e^x + Axy^2 + \mu$ where the exponential is strictly positive. (There would be a similar contradiction in (2) too.) So it is impossible that $x^2 + By^2 \neq C$.