Universitetet i Oslo
Økonomisk institutt

ECON3120/4120 – Mathematics 2, spring 2007

Answers to problems handed out on 15 January

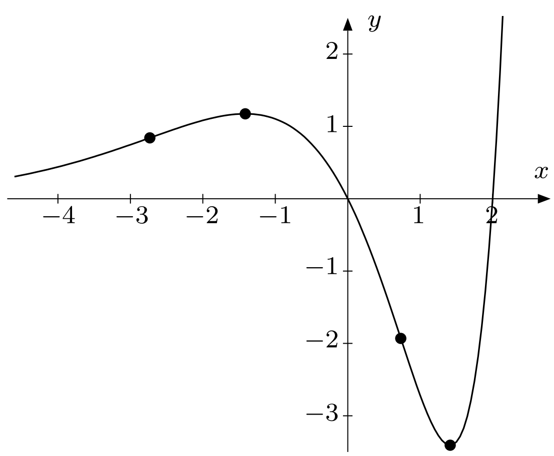
Problem 1

(a)
$$f'(x) = (2x-2)e^x + (x^2-2x)e^x = (x^2-2)e^x,$$
$$f''(x) = 2xe^x + (x^2-2)e^x = (x^2+2x-2)e^x.$$

(b) We can factor $f(x)$ as $f(x) = x(x-2)e^x$. Since a product of two or more factors is 0 precisely when at least one factor is 0, and since $e^x \neq 0$ for all x, it follows that $f(x) = 0 \iff x = 0$ or $x = 2$. You can see from the figure below that the graph of f intersects the x-axis precisely at the two points corresponding to $x = 0$ and $x = 2$.

The stationary points of f are the solutions of $f'(x) = 0$, that is, the points $x = \pm\sqrt{2}$ ($\approx \pm 1.4142$). A sign diagram (= "fortegnsdiagram") for $f'(x)$ reveals that f is increasing in the interval $(-\infty, -\sqrt{2}]$, decreasing in $[-\sqrt{2}, \sqrt{2}]$, and increasing again in $[\sqrt{2}, \infty)$. Therefore $x_1 = -\sqrt{2}$ is a local maximum point for f and $x_2 = \sqrt{2}$ is a local minimum point. We could also see this by observing that $f''(x_1) < 0$ and $f''(x_2) > 0$.

The equation $f''(x) = 0$ has the roots $x = -1 \pm \sqrt{3}$ ($\approx -1 \pm 1.73205$). These two points are inflection points (= "vendepunkter") for f, since $f''(x)$ equals 0 there and changes sign around them. In the interval $(-\infty, -1-\sqrt{3}\,]$ we have $f''(x) \geq 0$, and so f is convex there. In $[-1-\sqrt{3}, -1+\sqrt{3}\,]$, f is concave because $f''(x) \leq 0$, and finally $f''(x) \geq 0$ for $x \geq -1+\sqrt{3}$, so f is convex in the interval $[-1+\sqrt{3}, \infty)$. Both the local extreme points and the inflection points are indicated by the corresponding points on the graph of f.



The graph of $f(x) = (x^2-2x)e^x$.

Problem 2

(a) $g(x)$ is defined for $x > -1$, that is, its domain of definition (= "definisjons-område") is the interval $(-1, \infty)$.

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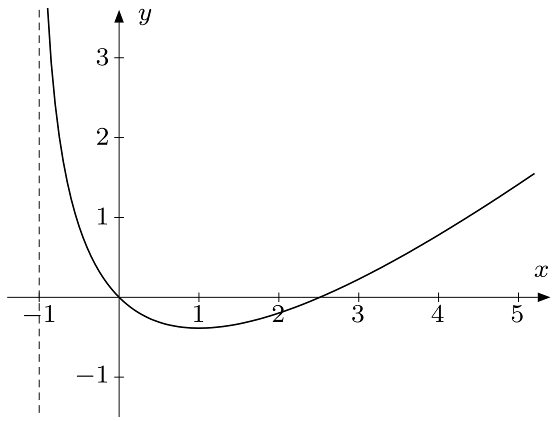
(b) $g'(x) = 1 - \dfrac{2}{x+1} = \dfrac{x-1}{x+1}$, $\quad g''(x) = \dfrac{2}{(x+1)^2}$.

(c) It is easy to see that $g'(x) \begin{cases} < 0 & \text{if } -1 < x < 1, \\ = 0 & \text{if } x = 1, \\ > 0 & \text{if } x > 1. \end{cases}$

It follows that g is strictly decreasing in $(-1, 1]$ and strictly increasing in $[1, \infty)$. Therefore $x = 1$ is a global minimum point for g. There are no other local or global extreme points. Since $g''(x) > 0$ for all $x > -1$, the function is convex over all of $(-1, \infty)$ and has no inflection points.



The graph of $g(x) = x - 2\ln(x+1)$.

The figure shows the graph of g. It is clear that $\lim_{x->(-1)^+} g(x) = \infty$, and the graph therefore has a vertical asymptote $x = -1$. It can also be shown that $\lim_{x\to\infty} g(x) = \infty$. To show that, write the function as $g(x) = x\Big(1 - \dfrac{2\ln(x+1)}{x}\Big)$ and show that the expression inside the big parenthesis tends to 1. (Use l'Hôpital's rule for ∞/∞.)

Problem 3

$h(x, y) = \ln(y - x^2)$ is defined when $y > x^2$, that is, at all points above (but not on) the parabola $y = x^2$ in the xy-plane.

Problem 4

If $\dfrac{(x+2)\ln(1+x)}{x-2} = 0$, then we must have $(x+2)\ln(1+x) = 0$. A product is equal to 0 if and only if at least one factor is 0, so we must have either $x+2 = 0$ or $\ln(1+x) = 0$, i.e. $x = -2$ or $x = 0$. However, if $x = -2$ we get $1+x = -1 \le 0$, and then $\ln(1+x)$ is not defined, so the left-hand side of the given equation simply does not make sense. Hence, the equation has exactly one solution, namely $\underline{x = -1}$.

Now, you didn't have $x = 2$ as a solution, did you? I hope not. Some people seem to believe that $A/B = 0$ if and only if $A = 0$ or $B = 0$. But this is wrong, of course. If $B = 0$, then the expression A/B does not make sense at all.