ECON 4120 - 03.03.14

Optimization with multiple variables

Necessary conditions

A derivable function $f(x,y)$ only have an optima in its interior if the F.O.C. is satisfied at a point,

$$f_1'(x,y)=0 \qquad f_2'(x,y)=0,$$

that is it has a stationary point

Sufficient condition for global optima

Assume (x_0,y_0) is a stationary point of a C^2 function $f(x,y)$ in the interior of a convex set S in the xy-plane.

- If for all $(x,y)\in S$

  $$f_{11}''(x,y)\le 0,\ f_{22}''(x,y)\le 0,\ f_{11}''(x,y)\,f_{22}''(x,y)-[f_{12}''(x,y)]^2\ge 0$$

  then (x_0,y_0) is a maximum for $f(x,y)$ in S.

- If for all $(x,y)\in S$

  $$f_{11}''(x,y)\ge 0;\ f_{22}''(x,y)\ge 0,\ f_{11}''(x,y)\,f_{22}''(x,y)-[f_{12}''(x,y)]^2\ge 0$$

  then (x_0,y_0) is a minimum for $f(x,y)$ in S

In general, when $f_1'(x_0,y_0)=f_2'(x_0,y_0)=0$, then (x_0,y_0) has one of the following properties:

- local maximum
- local minimum
- local saddle point

Sufficient condition for local optima

Let $f(x,y)$ be a C^2-function in S, and let (x_0,y_0) be a stationary point in the interior of S.
Let $A = f''_{11}(x_0,y_0)$, $B = f''_{12}(x_0,y_0)$, $C = f''_{22}(x_0,y_0)$

Then we have

- $A<0$, $AC-B^2>0$ $\Rightarrow$ (x_0,y_0) local minimum
  (and thus $C<0$)
- $A>0$, $AC-B^2>0$ $\Rightarrow$ (x_0,y_0) local maximum
  (and thus $C>0$)
- $AC-B^2<0$ $\Rightarrow$ (x_0,y_0) local saddle path
- $AC-B^2=0$ $\Rightarrow$ No decision from this test

<u>Proof:</u> Assume have ~~solution (maximum)~~ local maximum in (x_0,y_0).

Then define $g(t) = f(x_0+th, y_0+tk)$ where $h,k \in \mathbb{R}$ arbitrary.

Know $t=0$ must give local max, ~~...~~ sufficient that $g''(0)<0$ for max.

$g'(t) = f'_1(x_0+th, y_0+tk)h + f'_2(x_0+th, y_0+tk)k$

~~$g'(0) = A$~~ ~~$g'(0) = f'_1(x_0)h + f'_2$~~

$g''(0) = Ah^2 + 2Bhk + Ck^2 = A\left[\left(h+\frac{B}{A}k\right)^2 + \frac{AC-B^2}{A^2}k^2\right] < 0$

ok if $A<0$ and $[\cdot]>0$, which $AC-B^2>0$ ensures (sufficient). $\square$

<u>Example (global)</u>

$$\max_{x,y}\{\underbrace{-2x^2-2xy-2y^2+36x+42y-158}_{f(x,y)}\}$$

$\frac{\partial f}{\partial x} = -4x-2y+36=0$ $\qquad$ $\frac{\partial f}{\partial y} = -2x-4y+42=0$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad -x-2y+21=0$

$y=18-2x$
$-x-36+4x+21=0$
$3x=15$ $\underline{x=5}$ $\underline{y=8}$

$\frac{\partial^2 f}{\partial x^2} = -4$ $\quad \frac{\partial^2 f}{\partial x \partial y} = -2$ $\quad \frac{\partial^2 f}{\partial y^2} = -4$

$\frac{\partial^2 f}{\partial x^2} \le 0 \ \forall (x,y)$, $\frac{\partial^2 f}{\partial y^2} \le 0 \ \forall (x,y)$ $\quad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2 = (-4)^2 - (-2)^2 = 16-4 = 12 \ge 0 \ \forall (x,y)$

$\Rightarrow$ <u>Global maximum in $(x,y)=(5,8)$</u>

Example (local)

Examine the stationary points of

$f(x,y) = x^3 - x^2 - y^2 + 8$

$f'_x = 3x^2 - 2x = 0 \quad x(3x-2) = 0 \quad \underline{x=0}$ or $\underline{x = \frac{2}{3}}$

$f'_y = -2y = 0 \quad \underline{y=0}$

$f''_{xx} = 6x - 2 \quad f''_{xy} = 0 \quad f''_{yy} = -2$

↳ because of this, the test cannot give minimum

f''_{xx} depends on x, may be negative $\Rightarrow$ no global maximum from the test

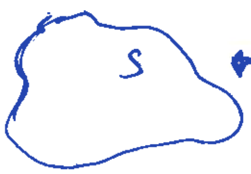
(x,y)	f''_{xx}	f''_{yy}	f''_{xy}	$f''_{xx} f''_{yy} - (f''_{xy})^2$	Point property
$(0,0)$	-2	-2	0	4	Local maximum
$(\frac{2}{3}, 0)$	2	-2	0	-4	Saddle point

Extreme value theorem

$$\boxed{\text{if } f(x,y) \text{ is a continuous function in a closed, bounded set } S \text{ in the plane, then there exists a point } (a,b) \text{ and a point } (c,d) \text{ such that } f(a,b) \le f(x,y) \le f(c,d) \text{ for all } x,y \in S}$$

~~such that~~

↳ all about existens, sufficient, not necessary.



← closed if containing all its boundary points

- How to find optimal points when you know they exist, over a closed bounded set:

  I Find the stationary points in the interior of S

  II Find the biggest and smallest value of f on the boundary, and which points gives these values.

  III Compare the values from I and II, the one giving biggest value is max, the one giving smallest is min.

Example

$f(x,y) = x^2 + y^2 + y - 1 \qquad S = \{(x,y) : x^2 + y^2 \leq 1\}$

I $\quad \frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x = 0$

$\frac{\partial f}{\partial y} = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2} \qquad \underline{(x,y) = (0,-\frac{1}{2})}$ in the exterior, only stationary point.

II $\quad x^2 + y^2 = 1$ ~~⇒ biggest possib~~

$\Rightarrow f(x,y) = x^2 + y^2 + y - 1 = y$

Biggest possible when $y = 1 \Rightarrow x = 0 \quad \underline{(x,y) = (0,1) \text{ biggest possible}}$

Smallest possible when $y = -1 \Rightarrow x = 0 \quad \underline{(x,y) = (0,-1) \text{ smallest possible}}$

III $\quad f(0,-\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} - 1 = \frac{1}{4} - \frac{2}{4} - \frac{4}{4} = \underline{-\frac{5}{4}}$

$f(0,1) = 1 + 1 - 1 = \underline{1}$

$f(0,-1) = 1 - 1 - 1 = \underline{-1}$

$\Rightarrow$ $\boxed{\begin{array}{l}\text{Maximum on boundary, } (x,y) = (0,1) \\ \text{Minimum in exterior, } (x,y) = (0,-\frac{1}{2})\end{array}}$