PROBLEM FROM FINAL LECTURE

Consider the problem

$$\max\{\ln(x+1)+\ln(y+1)\} \text{ s.t. } x+2y\le \frac{5}{2},\ x+y\le 2.$$

a) Set up the Kuhn-Tucker necessary conditions for candidates.

b) Find the best candidate(s)

c) Can you verify that the best candidate is a global maximum?

d) If the first constraint $c=\frac{5}{2}$ changes marginally, find $V'(c)$, where V is the value function of the problem.

Solution

a) $\mathcal{L}(x,y) = \ln(x+1) + \ln(y+1) - \lambda\left(x+2y-\frac{5}{2}\right) - \mu(x+y-2)$

$$\boxed{\begin{array}{ll} (1) & \dfrac{\partial \mathcal{L}}{\partial x} = \dfrac{1}{x+1} - \lambda - \mu = 0 \\[2ex] (2) & \dfrac{\partial \mathcal{L}}{\partial y} = \dfrac{1}{y+1} - 2\lambda - \mu = 0 \\[2ex] (3) & \lambda \geq 0 \quad (\lambda = 0 \text{ if } x+2y < \frac{5}{2}) \\[1ex] (4) & \mu \geq 0 \quad (\mu = 0 \text{ if } x+y < 2) \\[1ex] (5) & x+2y \leq \frac{5}{2} \\[1ex] (6) & x+y \leq 2 \end{array}}$$

b) I regard the four cases constraints (5) and (6) give.

- (5) and (6) binding:

$\underline{x+2y = \frac{5}{2}}$, $\underline{x+y=2}$ $\Rightarrow$ $y = 2-x$

$x + 4 - 2x = \frac{5}{2}$ $\quad \underline{x = \frac{3}{2}}$ $\quad \underline{y = \frac{1}{2}}$

(1) then gives $\underline{\mu = \frac{2}{5} - \lambda}$, (2) gives $\frac{2}{3} - 2\lambda = \frac{2}{5} - \lambda$

$\Rightarrow \lambda = \frac{2}{3} - \frac{2}{5} = \underline{\frac{4}{15}} \geq 0$ $\qquad \mu = \frac{2}{5} - \frac{4}{15} = \frac{6-4}{15} = \underline{\frac{2}{15}} \geq 0$

This satisfy $\lambda \geq 0$ and $\mu \geq 0$

$\Rightarrow$ $\boxed{(x,y) = \left(\frac{3}{2}, \frac{1}{2}\right) \text{ is a candidate}}$

- (5) binding, (6) non-binding

$\underline{x+2y=\frac{5}{2}}$, $\underline{\mu=0}$

$\underline{x=\frac{5}{2}-2y}$ (1) then gives $\lambda = \frac{1}{\frac{5}{2}-2y+1} = \underline{\frac{1}{\frac{7}{2}-2y}}$

(2) then gives $\frac{1}{1+y} = \frac{2}{\frac{7}{2}-2y} \Rightarrow \frac{7}{2}-2y = 2+2y$

$4y=\frac{3}{2}$ $\underline{y=\frac{3}{8}}$ $\Rightarrow$ $\underline{x=\frac{7}{4}}$ This violates (6), because $x+y=\frac{3}{8}+\frac{7}{4}=\frac{17}{8}>2$

$\Rightarrow$ This case is impossible

- (5) non-binding, (6) binding

$\underline{\lambda=0}$ $\underline{x+y=2}$ $\Rightarrow$ $\underline{x=2-y}$

(1) gives $\mu = \frac{1}{x+1} = \underline{\frac{1}{3-y}}$

(2) gives $\frac{1}{1+y} = \frac{1}{3-y} \Rightarrow 3-y=1+y$ $2y=2$ $\underline{y=1}$

$\Rightarrow$ $\underline{x=1}$ This violates condition (5), as

$x+2y = 1+2=3=\frac{6}{2}>\frac{5}{2}$ $\Rightarrow$ This case is impossible

- (5) non-binding, (6) non-binding

$\underline{\lambda=0}$ $\underline{\mu=0}$ $\Rightarrow$ (1) gives $\frac{1}{x+1}=0$, but this is impossible, as $1\neq 0$. $\Rightarrow$ This case is impossible

Since we are left with only one candidate, $(x,y) = (\frac{3}{2}, \frac{1}{2})$, it follows that this is the best candidate.

$\Rightarrow$ $\underline{(x,y) = (\frac{3}{2}, \frac{1}{2}) \text{ is the best candidate}}$

c) Let's check if $\mathcal{L}(x,y)$ is concave (observe that the extreme value theorem does not work here, as the two constraints does not make a closed and bounded set for x and y).

$$\mathcal{L}'_x = \frac{1}{x+1} - \lambda - \mu \Rightarrow \mathcal{L}''_{xx} = \underline{-\frac{1}{(x+1)^2}} \leq 0$$

$$\mathcal{L}'_y = \frac{1}{y+1} - 2\lambda - \mu \Rightarrow \mathcal{L}''_{yy} = \underline{-\frac{1}{(y+1)^2}} \leq 0$$

$$\mathcal{L}''_{xy} = \underline{0} \Rightarrow \mathcal{L}''_{xx}\mathcal{L}''_{yy} - (\mathcal{L}''_{xy})^2 = \underline{\frac{1}{(x+1)^2(y+1)^2}} \geq 0$$

This is true for all relevant x and y. Notice that actually we demand $x > -1$ and $y > -1$ in this exercise, or the $\ln(x+1)$/ $\ln(y+1)$ is not defined.

We see that $\mathcal{L}(x,y)$ is concave over all relevant (x,y)

$\Rightarrow$ $\underline{\text{Our best candidate is a global maximum}}$

(Thus, it indeed does solve the problem).

d) Remember that when a constraint of the form $g(x,y) \le c$ has a change (increase) in c, then the marginal change in the value function equals the multiplier corresponding to that constraint.

Thus, here we get $V'(c) = \lambda$, where λ is found from the case where (5) and (6) are binding, since this case gave the best candidate.

$$\Rightarrow \underline{\underline{V'(c) = \lambda = \frac{4}{15}}}$$

As a remark, if you want to find the approximate change in the value function V when $x+2y \le \frac{5}{2}$ changes to $x+2y \le \frac{5}{2} \pm 0{,}2$, then we have

$$\Delta V \approx V'(c)\,\Delta c = \frac{4}{15} \cdot (\pm 0{,}2) = \underline{\underline{\pm \frac{4}{75}}}$$

(+ when $\Delta c = +0{,}2$, − when $\Delta c = -0{,}2$).