Max / min (& friends...)

Some basics

If f is a function (that returns $f(x) = y \in \mathbb{R}$) then a <u>maximum</u> for f [also called: a <u>global</u> maximum] is an x^* such that

$$f(x^*) \geq f(x), \quad (\text{all } x \neq x^*)$$

It is <u>strict</u> if $f(x^*) > f(x)$, all $x \neq x^*$.

Note: it is the <u>input</u>. The greatest output $f(x^*)$ is called the <u>maximum value</u>.

Minimum, strict minimum: analogous.

x^* is an <u>extremum</u> if it is a maximum or a minimum.

Then: $f(x^*)$ an <u>extreme value</u>.

Max / min cont'd

The concept allows f to be defined on whatever set we may wish, e.g., {bicycle, car, train}

This course: focus on subsets of $\mathbb{R}$ ← (one real variable x)
or $\mathbb{R}^2$ (two real var's (x, y)) and a little bit on $\mathbb{R}^n$ for general n.

Then we can talk about <u>local</u> max / min:

- Recall that subset U of $\mathbb{R}^n$ is <u>open</u> if it contains none of its boundary points.
- A point P [x^* or (x^*, y^*) or $(x_1^*, \ldots, x_n^*)$] is a <u>local maximum</u> for f if:

→ there exists some open U centered at P such that

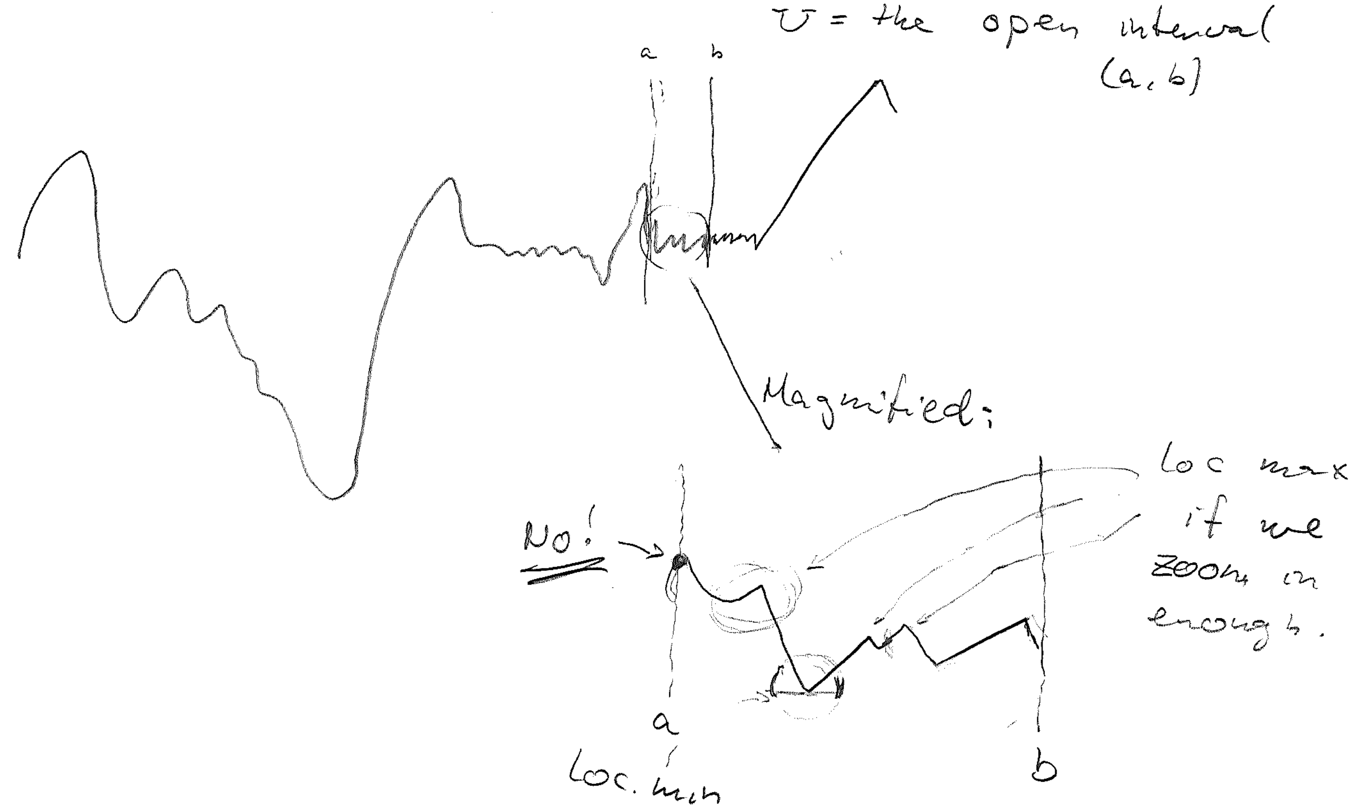
→ we can zoom in on U and <u>discard</u> everything else (let $g = f$ on U, <u>undefined otherwise</u>)

→ and then get a maximum.

Strict loc. max, loc min, strict loc. min, loc. extrema ...

Max/min: local

Like this ($n=1$):



Why the "open" set?
See the "No!" point. Need "proper inside".

(Local) max/min: questions

Q1: <u>What</u> could they possibly look like?

Q2: How to <u>find</u> possible candidates for max/min?

Q3: How to <u>decide</u> once one is found, whether it <u>is</u> a local max/min
– and if so, a <u>global</u> as well?

But.... waitaminute ....

Q0: Can we even know that one <u>exists</u>?

$f(x) = x$, defined for all x? No!

$f(x) = x$, defined for $x \in [-5, 2]$? Yes!

$\rightsquigarrow$ The extreme value theorem.

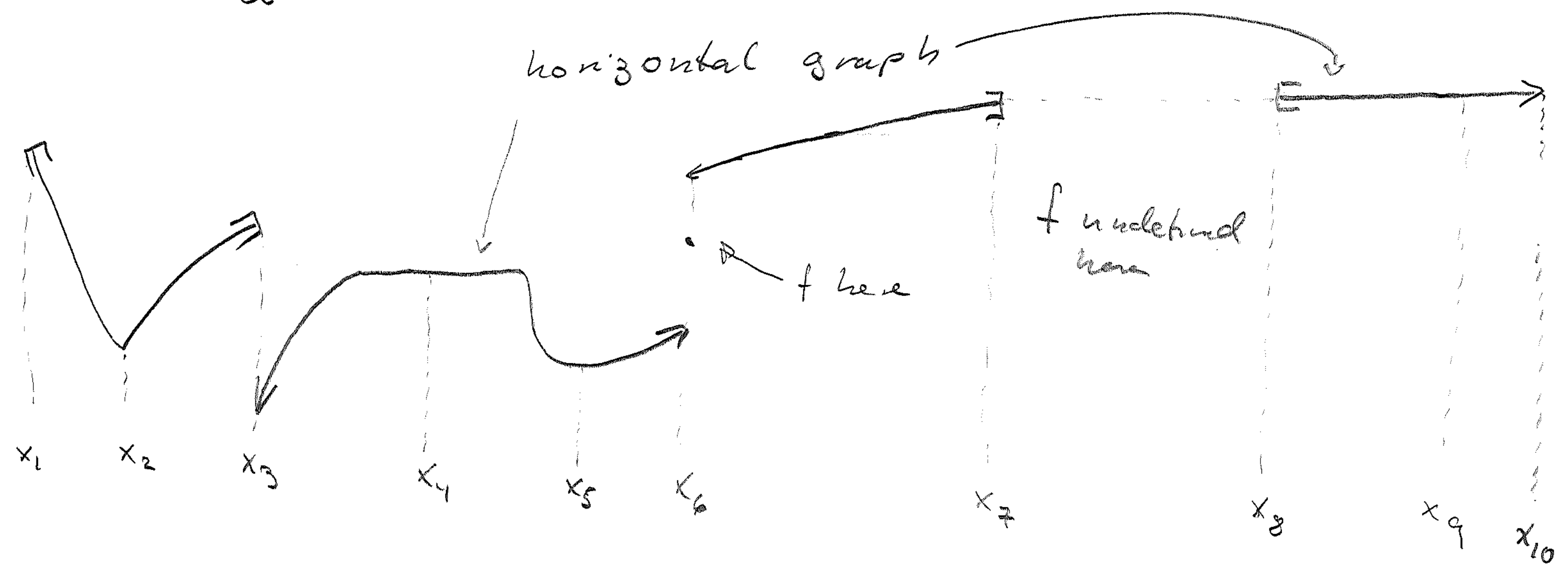
(We will get to it.)

A1: what they could look like

• Discontinuity points	Not important in Math 2
• Nondifferentiability points	You must know that $\lvert x \rvert$ has a minimum at $x = 0$
• Stationary points (where $f'(x) = 0$; $\frac{\partial f}{\partial x_i}(x_1 \dots x_n) = 0 \quad \forall i$)	Essential!!!
• Boundary points ↳ endpoints when $n = 1$	Quite a lot of attention, especially for $n > 1$.

$n=1$, illustration

means: $f(x^*) = \lim_{x \to a^-} f(x)$



Which ones are local max? Local min?

Global max? Global min?

<u>Strictly</u> so?

$n = 1$

A3: how to decide

a

* Left endpoint: f defined at a.

$f' > 0$ at $a \Rightarrow$ loc. min

$f' < 0$ at $a \Rightarrow$ loc. max

$f' = 0$ at a .... ?

Depends on what happens just to the right of a.

* Right endpoint: switch sign! (why?)

* Stationary points:

  → first-derivative test

  → second-derivative test.

$n=1$, A3 cont'd

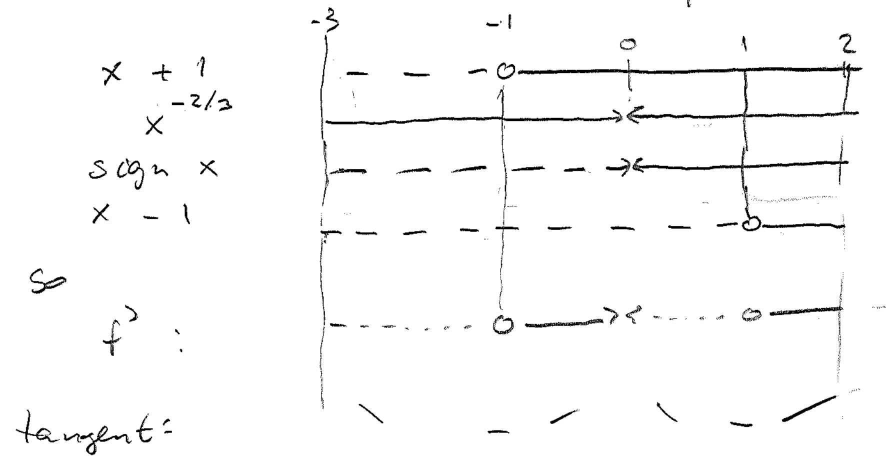
* Stationary points. The first-derivative test

Ex.: If $f(x) = |x|^{7/3} - 7|x|^{1/3}$, $x \in [-3, 2]$

$$f'(x) = \frac{7}{3}(x-1)(x+1) \cdot x^{-2/3} \cdot \operatorname{sign} x$$

$\operatorname{sign} x = \frac{d}{dx}|x|$, $x \neq 0$.

as a product.



$x=-1$: loc. min. $x = 1$: loc. min.

$x = -3$, $x = 0$ (a cusp!) and $x=2$: loc max.

Q: Global ...?

* The second-derivative test: later!

$n > 1$:

* Boundary points:
  Not so easy anymore!
  · An interval has two endpoints.
  · A set like the first quadrant $\{(x,y);\ x \geq 0,\ y \geq 0\}$
  has infinitely many!

This is why we have a theory for constrained max/min!

→ Lagrange's method

→ Kuhn–Tucker cond's.

* Stationary points:
  First-derivative test ...?
  Second-derivative test!

* Second derivatives,
  concavity/convexity
  max/min :

Time to take notes!

$f'' > 0$?	f''_{xx} and f''_{yy} and $f''_{xx} f''_{yy} - (f''_{xy})^2$
$f'' \geq 0$?	> 0 ?
	≥ 0 ?