UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Term paper in: ECON3120/4120 – Mathematics 2

Handed out: 3.3.2017

To be delivered by: 23.3.2017 – within 15.00.

Place of delivery: Fronter

Further instructions:

- This term paper is **compulsory**. Candidates who have passed the compulsory term paper in a previous semester, do not have the right to hand in the term paper again. This also applies if the candidate did not pass the exam.
- You must deliver the term paper within the given deadline. Term papers delivered after the deadline, **will not be corrected**.*
- All term papers must be delivered as stated above. Do not deliver your term paper to the course teacher or send it by e-mail.
- The term paper will not be given a grade that influences the final grade in the course.
- **Note:** The students can feel free to discuss with each other how to solve the problems, but each student is supposed to formulate her/his own answers. Only single-authored papers are accepted, and papers that for all practical purposes are identical will not be approved.
- Information about citing and referring to sources: **http://www.uio.no/english/studies/about/regulations/sources/**
- Information about consequences of cheating: **http://www.uio.no/english/studies/admin/examinations/cheating/index.html**
- If the term paper is not accepted, you will be given a new attempt (this will not apply if you deliver a blank page). If you still do not succeed, you will not be permitted to take the exam in this course. You will then be withdrawn from the exam.

*) If a student believes that she or he has a valid reason not to meet the deadline, she or he must seek a formal extension from the Department of Economics. Read more about this here: http://www.uio.no/english/studies/admin/compulsory-activities/sv-absence-from-compulsory-tuition-activities.html

ECON3120/4120 Mathematics 2

Compulsory term paper, spring term 2017.

There are 3 pages of problems to be solved, not counting this page.

Justifying answers:

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Minimum requirements to pass this assignment:

- You will pass if *each of problems 1, 2, and 3* is scored as good enough to pass, if it were judged to be one exam stand-alone.
  (The commonly applied pass mark in mathematics is forty percent, and this course by default uses uniform weighting over letter-enumerated items.)
- If you fail one problem despite a decent attempt at it, we may still let you pass upon judging the overall quality of the paper. In particular, we shall take into account that two letter items are marked as demanding.

The paper does not count towards your final course grade!

- Passing the term paper is required in order to sit in on the exam (see the Department's rules). Other than that, it does not in any way count towards your grade, and the exam grading committee will not see your term paper.

Problem 1

(a) Let $h(x) = x^e \cdot [e^{sx} + e^{s^3 x}]$ and $g(x) = \frac{\ln(1+h(x))}{h(x)}$. Consider for each $s \neq 0$ the limits

$$\lim_{x \to 0^+} g(x) \qquad \text{and} \qquad \lim_{x \to +\infty} g(x)$$

For each $s \neq 0$: what does l'Hôpital's rule tell you about each of these limits?

(b) For s in a certain range, the function $h(x)$ of part (a) will have a global maximum point $x^* > 0$. The maximum value $V = h(x^*)$ depends on s. Find an expression for $V'(s)$.

(c) Let $u(x) = x - x^x$ and $v(x) = \log_x(1 - u(x))$. Find $v'(x)$.
(*Hint:* $x^{v(x)} = 1 - u(x)$.)

(d) An account accumulates interest as $e^{\rho t}$ continuously compounded, where t is time measured in years.

- Suppose first that $\rho = 0.025$. Find p so that the annual interest rate is $p\,\%$.
- Suppose instead that the annual interest rate is $\pi\,\%$ (where π is the constant ≈ 3.14159). Convert this to a continuously compounded ρ.

(e) *[This could be demanding]* Let f be a twice continuously differentiable function defined on $(0, \infty)$, and assume f is convex and that $\lim_{x \to 0^+} f(x) < 0$. Try to formulate a *proof by contradiction* that f cannot have two (nor more) zeroes.

Problem 2 Let $f(x,y) = e^{x^3 y} - xy - 1$.

(a) i) f has the $f(x,y) = 0$ when $xy = 0$. Use this property, and the behaviour of the functions $g(z) = f(\sqrt{z}, \sqrt{z})$ and $h(z) = f(\sqrt{z}, -\sqrt{z})$ to

- show that f has a stationary point at $(0,0)$ *without* calculating the general expressions for $f'_x(x,y)$ nor $f'_y(x,y)$.
- classify this stationary point *without* calculating second derivatives of f (hint: what are $g'(0^+)$ and $h'(0^+)$?)

ii) Now, calculate the partial second derivatives of f and check whether the second-derivative test can classify the stationary point $(0,0)$.

Let from now on $t > 0$ be a constant, and consider the problem

$$\max\ f(x,y) \qquad \text{subject to} \qquad (tx+1)y \le 1, \qquad x \ge 0 \qquad \text{and} \quad y \ge 0 \tag{P}$$

(b) i) State the Kuhn–Tucker conditions associated with problem (P).

ii) Do we know already, without further calculations, that there will be at least one point which satisfies these conditions?

(c) Consider the points such that $x = 0$ and $0 < y < 1$. Which of these points – if any – will satisfy the Kuhn–Tucker conditions?

(d) It can be shown – but you are not asked to do so – that if the Kuhn–Tucker conditions hold with $x > 0$ and $(tx+1)y = 1$, then

$$\left(3x^2 - \frac{t}{tx+1}\right)e^{x^3/(tx+1)} = \frac{1}{(tx+1)^2}$$

Show that there exists a positive x satisfying this equation.

Problem 3

(a) Let k be a constant. Show that

$$\int ((k+2)t^2 - 2t + 1)k^3 e^{kt}\, dt = C + \frac{k^2(k+2)t^2 - 4k(k+1)t + (k+2)^2}{e^{-kt}}$$

(b) Find the general solution of the differential equation

$$\dot{y}(t) = (1-2t)e^{-2t} \cdot y(t)$$

(*Hint:* Use part (a) with $k = -2$.)

(c) Find the particular solution satisfying $z(2) = 3$ of the differential equation

$$\dot{z}(t) = (1-2t)e^{-t} + z(t)$$

(d) *[This could be demanding]* It is a fact – and you shall not show – that there is a unique particular solution w of

$$\dot{w}(t) = 1 + \big(w(t)\big)^2 \quad \text{for } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \text{such that} \quad w(0) = 0.$$

Do not try to solve this differential equation, as it requires knowledge beyond this course.

- Show that w has an inverse τ, and find τ'.
