Left over from last time.

$$\vec{A}\vec{C} + \vec{B}\vec{C} = (\vec{A} + \vec{B})\vec{C}.$$

$$\vec{A}\vec{C} + \beta\vec{C} = (\vec{A} + \beta\vec{I})\vec{C}.$$

Transpose.

Ex. $\underset{3\times 2}{\vec{A}} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 0 \end{pmatrix} \xrightarrow{\text{transpose}} \underset{2\times 3}{\vec{A}' / \vec{A}^T} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \end{pmatrix}$

Def. $\underset{m\times n}{\vec{A}} = (a_{ij}) \xrightarrow{\text{transpose}} \underset{n\times m}{\vec{A}'} = (a_{ij}')$

we have $\cancel{a_{ij} = a'}$ $a_{ij}' = a_{ji}$

Rules: $(\vec{A}')' = \vec{A}.$

$(\vec{A}+\vec{B})' = \vec{A}' + \vec{B}'.$

$(\alpha\vec{A})' = \alpha\vec{A}'$

$(\vec{A}\vec{B})' = \vec{B}'\vec{A}'.$

Verify: (c_{ij}) (d_{ij})

c_{ij} = jth row of $\vec{A}$ · ith column of $\vec{B}$ = d_{ij} = ith column of $\vec{B}$ · jth row of $\vec{A}$.

<u>Symmetric matrices</u>

Ex. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 4 \\ 0 & 4 & 2 \end{pmatrix}$ observe that: $a_{ij} = a_{ji}$.

$\vec{A}$ is symmetric $\Longleftrightarrow$ $\vec{A}' = \vec{A}$

Implications:

1. square matrices.
2. symmetric about the main diagonal. $\Leftrightarrow$ symmetric.

Ex. $\vec{X}$ is $m\times n$, prove that $\vec{X}'\vec{X}$ and $\vec{X}\vec{X}'$ are symmetric

Proof. $(\vec{X}'\vec{X})' = (\vec{X}')((\vec{X}')') = \vec{X}'\vec{X}$.

$(\vec{X}\vec{X}')' = ((\vec{X}')')(\vec{X}') = \vec{X}\vec{X}'$.

Gaussian Elimination.

$x_1 + x_2 = 1 \quad (1)$

$-x_1 + x_2 = 0. \quad (2)$

Augmented coefficient matrix.

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 0 \end{array}\right)$$

Solve by substitution: from (2) we have $\cancel{x_1}$ $x_2 = x_1$.

substitute $x_2 = x_1$ into (1) $\Rightarrow$ $\cancel{x_1+x_2=}$ $x_1 + x_1 = 1 \Rightarrow x_1 = \frac{1}{2}$.

$x_2 = \frac{1}{2}$.

Solve by elimination: add (1) to (2)

$\Rightarrow$ $0x_1 + 2x_2 = 1$, $(2')$

$$\Rightarrow \begin{array}{ll} x_1 + x_2 = 1 & (1) \\ 2x_2 = 1 & (2') \end{array}$$

$$\begin{array}{l} x_1 + x_2 + x_3 = 6 \\ x_1 + \quad + 2x_3 = 7 \\ x_1 + 4x_2 + x_3 = 12 \end{array} \quad \begin{array}{cc} -1 & -1 \end{array} \quad \sim \quad \begin{array}{l} x_1 + x_2 + x_3 = 6 \\ -x_2 + x_3 = 1 \\ 3x_2 \quad = 6 \end{array} \quad 3$$

$$\sim \begin{array}{l} x_1 + x_2 + x_3 = 6 \\ -x_2 + x_3 = 1 \\ 0 + 3x_3 = \cancel{10}\,9 \end{array}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 0 \end{array}\right) \begin{array}{c} 1 \\ \hookleftarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 0 & 2 & 7 \\ 1 & 4 & 1 & 12 \end{array}\right) \begin{array}{cc} -1 & -1 \\ \hookleftarrow & \\ & \hookleftarrow \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & 0 & 6 \end{array}\right) \begin{array}{c} \\ 3 \\ \hookleftarrow \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \rightarrow \text{upper triangular.}$$

leading entries.

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 2 & 1 & 1 & 7 \\ 2 & -4 & 4 & 6 \\ 3 & 4 & 0 & 11 \end{array}\right) \begin{array}{ccc} -2 & -2 & -3 \\ \hookleftarrow & & \\ & \hookleftarrow & \\ & & \hookleftarrow \end{array} \sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -5 & 3 & -1 \\ 0 & -10 & 6 & -2 \\ 0 & -5 & 3 & -1 \end{array}\right) \begin{array}{cc} & \\ -2 & -1 \\ \hookleftarrow & \\ & \hookleftarrow \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -5 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \begin{aligned} x_1 + 3x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= -1 \end{aligned}$$

2 equations, 3 unknowns
$\Rightarrow$ 1 degree of freedom.

$x_1 + 3x_2 = x_3 + 4$

$-5x_2 = -3x_3 - 1$.

$\Rightarrow \quad x_1 = -\frac{4}{5}x_3 + \frac{1}{5}$

$x_2 = \frac{3}{5}x_3 + \frac{1}{5}$.

$(x_1, x_2, x_3) = \left(-\frac{4}{5}t + \frac{1}{5},\ \frac{3}{5}t + \frac{1}{5},\ t\right)$ where t is any real number

Ex. ~~$x_1 + 2x_3$~~

$x_1 + 2x_2 + 3x_3 = 6 \quad \Rightarrow$ 2 degrees of freedom.

$x_1 = 6 - 2x_2 - 3x_3$

$(x_1, x_2, x_3) = (6 - 2t - 3s, t, s)$ s, t are any real number.

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 2 & 1 & 1 & 7 \\ 2 & -4 & 4 & 6 \\ 3 & 4 & 0 & 12 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -5 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$\Rightarrow$ no solution.

$\vec{A}\vec{x} = \vec{0} \Rightarrow$ always have at least one solution: $\vec{x} = \vec{0}$.

If there are solution where $\vec{x} \neq \vec{0}$, then you have infinitely many solutions.

Proof. If $\vec{x} \neq \vec{0}$ solves $\vec{A}\vec{x} = \vec{0}$.

Then $\vec{A}(\alpha\vec{x}) = \alpha\vec{A}\vec{x} = \alpha(\vec{A}\vec{x}) = \alpha\vec{0} = \vec{0}$.

when α can be any real number

$\vec{A}\vec{x} = \vec{0}$ has $\begin{cases} \text{one solution} \\ \text{or} \\ \text{infinitely many solutions.} \end{cases}$

$ax = b. \Rightarrow x = \cancel{a^{-1}b}\ \frac{b}{a}.$

$\vec{A}\vec{x} = \vec{b}. \Rightarrow \cancel{\vec{x} = \frac{\vec{b}}{\vec{A}}}$

if there exists such a matrix $\underline{\vec{A}^{-1}}$ that $\vec{A}^{-1}\vec{A} = \vec{I}$.

$\searrow$ Inverse of $\vec{A}$.

$\Rightarrow \vec{A}^{-1}\vec{A}\vec{x} = \vec{A}^{-1}\vec{b}.$

$\Rightarrow \quad \vec{x} = \vec{A}^{-1}\vec{b}.$