Determinant of order 2.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\vec{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\left(\begin{array}{cc:c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right) -\frac{a_{21}}{a_{11}} \sim \left(\begin{array}{cc:c} a_{11} & a_{12} & b_1 \\ 0 & a_{22}-\frac{a_{12}a_{21}}{a_{11}} & b_2 - \frac{b_1 a_{21}}{a_{11}} \end{array}\right)$$

$$\sim \left(\begin{array}{cc:c} a_{11} & a_{12} & b_1 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}} & \frac{a_{11}b_2 - a_{21}b_1}{a_{11}} \end{array}\right)$$

$$\Rightarrow x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22}-a_{12}a_{21}} \text{ ——— } \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

$$x_1 = \frac{b_1 - a_{12}x_2}{a_{11}} = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \text{ ——— } \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

$a_{11}a_{22}-a_{12}a_{21}$ must be non-zero for the problem to have a unique solution.

$D = a_{11}a_{22}-a_{12}a_{21}$ is called the determinant of $\vec{A}$.

denoted by $\det(\vec{A})$ or $|\vec{A}|$.

$$|\vec{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of order 3

$$\vec{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|\vec{A}| = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|\vec{A}|}$$

short version of Cramer's rule:

$$\vec{A}\vec{x} = \vec{b}.$$
$$n\times n \quad n\times 1 \quad n\times 1$$

provided that $|\vec{A}| \neq 0$

ith column

$$x_i = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1i} & b_1 & \cdots & \cancel{a_{n1}}\ a_{1n} \\ \vdots & \ddots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & b_n & \cdots & a_{nn} \end{vmatrix}}{|\vec{A}|}$$

As long as $|\vec{A}| \neq 0$, $\begin{cases} \vec{A}\vec{x} = \vec{b} \text{ has unique solution.} \\ \vec{A}\vec{x} = \vec{0} \text{ has unique solution } \vec{x} = \vec{0}. \end{cases}$

Lot of terms in the determinant.

In each term we have only one element from each row and each column. (1 guy from 1st row, 1 guy from the 2nd row...)

$\underset{n\times n}{|\vec{A}|}$ has $n\times(n-1)\times(n-2)\cdots 1 = n!$ terms.

look at $\underset{3\times 3}{\det(\vec{A})}$:

$$|\vec{A}| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) \qquad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+\, a_{12}(a_{23}a_{31} - a_{21}a_{33}) \qquad \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+\, a_{13}(a_{21}a_{32} - a_{22}a_{31}) \qquad \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|\vec{A}| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+\, (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

↘ cofactor expansion along row 1.

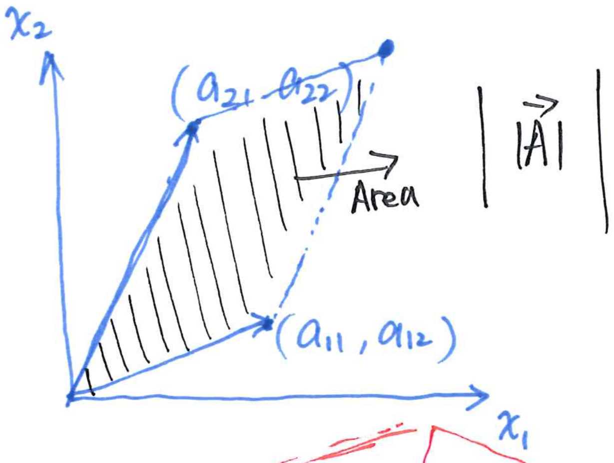
$$|\vec{A}| = (-1)^{3} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{4} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$-\, a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

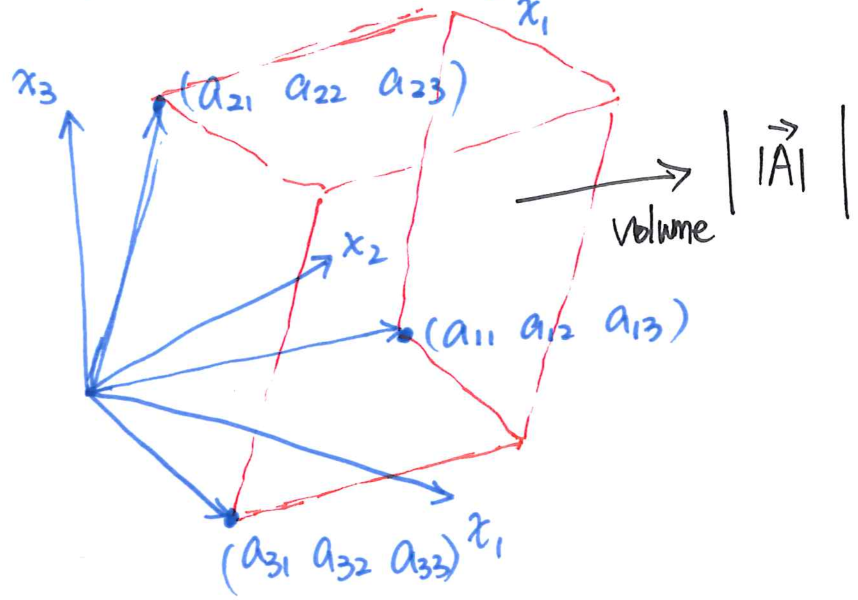
↘ along 2nd column.

Geometric interpretation.

$$\vec{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$



$$\vec{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\vec{I}_{n\times n}| = 1$$

$$|\vec{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

upper triangular.

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

lower triangular

$$|\vec{A}\,\vec{B}| = |\vec{A}||\vec{B}|$$

$$|\vec{A}+\vec{B}| \neq |\vec{A}|+|\vec{B}|$$

Inverse.

$\vec{A}\vec{x} = \vec{b}$.

$\underbrace{\vec{A}^{-1}\vec{A}}_{\vec{I}}\vec{x} = \vec{A}^{-1}\vec{b} \Rightarrow \vec{x} = \vec{A}^{-1}\vec{b}$.

$\vec{A}\vec{X} = \vec{X}\vec{A} = \vec{I}$. $\Rightarrow$ $\vec{X}$ and $\vec{A}$ are inverse of each other

$\vec{A}$ (and $\vec{X}$) are invertible.

$\Rightarrow$ only square matrices could have inverse.

$$\left(\begin{array}{c:c} \vec{A} & \vec{I} \\ n\times n & n\times n \end{array}\right) \xrightarrow{\text{Gaussian elimination}} \left(\begin{array}{c:c} \vec{I} & \vec{A}^{-1} \end{array}\right)$$

if $\vec{A}^{-1}$ exists $\Longleftrightarrow |\vec{A}| \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix}1\\2\end{pmatrix}\times 2 + \begin{pmatrix}0\\3\end{pmatrix}\times 1 & \begin{matrix}4\\17\end{matrix} \end{pmatrix} \quad \text{by columns}$$

$$= \begin{pmatrix} 1\times 2 + 0\times 1 & 1\times 4 + 0\times 3 \\ & \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 2 + 0\times 1 & 1\times 4 + 0\times 3 \\ 7 & 17 \end{pmatrix} \quad \text{by rows}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \overset{-1}{\llcorner} \sim \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

→ Gaussian

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$