Rules for determinants.

$\vec{A}$ is $n \times n$

- 1 If all elements in a row (column) are 0, then $|\vec{A}| = 0$.
- 2 $|\vec{A}'| = |\vec{A}|$

Elementary row operations

- 3 Multiply all elements in a row (column) by $\alpha \Rightarrow |\vec{A}|$ is also multiplied by α.
- 4 If two rows (columns) of $\vec{A}$ are interchanged, the determinant changes sign.
- 5 If we multiply a row (a column) by a factor α and add it to another row (another column), $|\vec{A}|$ is unchanged.
- 6 If two of the rows (columns) of $\vec{A}$ is proportional $\Rightarrow |\vec{A}| = 0$

Ex $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{vmatrix} \overset{\text{Rule 5}}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} \overset{\text{Rule 1}}{=} 0$.

comparing to real numbers and absolute value.

$|\vec{A}\vec{B}| = |\vec{A}||\vec{B}|$ $\qquad |\alpha\beta| = |\alpha||\beta|$

$|\alpha\vec{A}| = \alpha^n |\vec{A}|$

$|\vec{A} + \vec{B}| \neq |\vec{A}| + |\vec{B}|$ $\qquad |\alpha + \beta| \neq |\alpha| + |\beta|$

$\vec{A}^{-1}$ exists $\Leftrightarrow |\vec{A}| \neq 0$. $\qquad \alpha^{-1}$ exists if $|\alpha| \neq 0$.

Expansion by cofactors. more formal.

$$|\vec{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$ ⇒ expansion of $|\vec{A}|$ along the i-th row.

$C_{i1}, C_{i2}, \dots C_{in}$ are the cofactors of $a_{i1}, a_{i2} \dots a_{in}$.

$$C_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & & & \vdots & & \vdots \\ \vdots & & & & & \\ a_{i1} & \cdots & & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \vdots \\ a_{n1} & \cdots & & a_{nj} & \cdots & a_{nn} \end{vmatrix} \cdot (-1)^{i+j}$$

expansion along the j-column:

$$|\vec{A}| = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}.$$

Ex. $$|\vec{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \underset{\text{the 2nd row}}{\overset{\text{expansion along}}{=\!=}} a_{21}\cdot(-1)^3 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\cdot(-1)^4 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\cdot(-1)^5 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

or simply $a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

Now think about: $$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$|\vec{A}|$ is $n\times n$

$$\cancel{a_{11}C_{i1}}\ a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \begin{cases} |\vec{A}| & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

$$a_{1i}C_{1j} + a_{2i}C_{2j} + \cdots + a_{ni}C_{nj} = \begin{cases} |\vec{A}| & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

Inverse

- $\vec{A}$ has <u>an</u> inverse $\Leftrightarrow$ $|\vec{A}| \neq 0$.

Proof. necessity: $\vec{A}\vec{A}^{-1} = \vec{I}$. $\Rightarrow$ $\cancel{|\vec{A}||\vec{A}^{-1}| = |\vec{I}|}$ $|\vec{A}\vec{A}^{-1}| = |\vec{I}|$

$\Rightarrow$ $|\vec{A}||\vec{A}^{-1}| = |\vec{I}| = 1$. $\Rightarrow$ $|\vec{A}| \neq 0$.

sufficiency: if $|\vec{A}| \neq 0$. we could construct $\vec{A}^{-1} = \cdots$

- The inverse of $\vec{A}$ is unique, if it exists:

Proof. Let's say $\vec{A}$ has two inverses: $\vec{X}$ and $\vec{Y}$.

$$\vec{Y} = \vec{I}\vec{Y} = (\vec{X}\vec{A})\vec{Y} = \vec{X}(\vec{A}\vec{Y}) = \vec{X}\vec{I} = \vec{X}.$$

Def. of Inverse.

$\vec{A}\vec{X} = \vec{X}\vec{A} = \vec{I}$ then, $\vec{A}$ and $\vec{X}$ are inverses of each other.

But in practice. either $\vec{A}\vec{X} = \vec{I}$ or $\vec{X}\vec{A} = \vec{I}$ suffices for $\vec{A}$ and $\vec{X}$ to be inverse of each other.

Proof (Part): Let's say $\vec{A}\vec{X} = \vec{I}$, then $|\vec{A}||\vec{X}| = |\vec{I}| = 1$,

then $|\vec{A}| \neq 0 \Rightarrow \vec{A}^{-1}$ exists.

Then $\vec{A}^{-1}\vec{A}\vec{X} = \vec{A}^{-1}\vec{I} \Rightarrow \vec{X} = \vec{A}^{-1}$.

Ex. find the inverse of $\vec{A}$ if $\vec{A} - \vec{A}^2 = \vec{I}$.

$\vec{A}(\vec{I} - \vec{A}) = \vec{I}$. $\Rightarrow$ $\vec{I} - \vec{A} = \vec{A}^{-1}$.

Properties.

$\vec{A}$ and $\vec{B}$ are invertible $n \times n$

(1) $\vec{A}^{-1}$ is invertible. $(\vec{A}^{-1})^{-1} = \vec{A}$.

(2) $\vec{A}\vec{B}$ is invertible $(\vec{A}\vec{B})^{-1} = \vec{B}^{-1}\vec{A}^{-1}$.

(3) $\vec{A}'$ is invertible. $(\vec{A}')^{-1} = (\vec{A}^{-1})'$

(4) $(c\vec{A})^{-1} = \frac{1}{c}\vec{A}^{-1}$, whenever $c \neq 0$, real number.

The General formula for inverse.

$\vec{A}$ is invertible. ($|\vec{A}| \neq 0$)

$$\vec{A}^{-1} = \frac{1}{|\vec{A}|} \cdot \underbrace{\mathrm{adj}(\vec{A})}_{\text{the adjugate matrix of } \vec{A}}$$

$$\mathrm{adj}(\vec{A}) = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & \cdots & & \vdots \\ \vdots & & & \vdots \\ C_{1n} & \cdots & & C_{nn} \end{pmatrix}$$

Or in human language. substitute every element by its cofactor, then transpose the whole thing.

~~Prove~~ Proof. want to prove $\vec{A} \cdot \frac{1}{|\vec{A}|} \mathrm{adj}(\vec{A}) = \vec{I}$.

$$\frac{1}{|\vec{A}|} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & & & \vdots \\ \vdots & & & \vdots \\ C_{1n} & \cdots & & C_{nn} \end{pmatrix} = \begin{pmatrix} |\vec{A}| & 0 & \cdots & 0 \\ 0 & |\vec{A}| & \cdots & 0 \\ & & \ddots & \\ & & & |\vec{A}| \end{pmatrix} \frac{1}{|\vec{A}|} = \vec{I}.$$

Cramer's Rule. formally.

$\vec{A}\vec{x} = \vec{b}$. $n \times n$.

$$\vec{D}_j = \begin{pmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix}$$

$\downarrow$ j-th column.

$\vec{A}\vec{x} = \vec{b}$ has unique solution $\Leftrightarrow$ $\vec{A}$ is invertible / nonsingular ($|\vec{A}| \neq 0$)

$$x_1 = \frac{|\vec{D}_1|}{|\vec{A}|}, \quad x_2 = \frac{|\vec{D}_2|}{|\vec{A}|} \quad \cdots \quad x_n = \frac{|\vec{D}_n|}{|\vec{A}|}$$