Given $\vec{A}\vec{x}=\vec{b}$.

$n\times n$

(1) $|\vec{A}|\neq 0$. $\Rightarrow$ unique solution $\vec{x}=\vec{A}^{-1}\vec{b}$.

or $x_i=\frac{|\vec{D}_i|}{|\vec{A}|}$, Cramer's rule.

no matter what $\vec{b}$ is.

(2). $|\vec{A}|=0$. $\Rightarrow$ either no solution, or infinitely many solutions.

How to find out? by Gaussian Elimination.

Special case: $\vec{b}=\vec{0}$, $\vec{A}\vec{x}=\vec{0}$. $\Rightarrow$ Homogeneous systems of equtions.

$\vec{A}\vec{x}=\vec{0}$. always has a solution $\vec{x}=\vec{0}$ $\Rightarrow$ trivial solution.

Then when $\vec{A}$ is non singular ($|\vec{A}|\neq 0$) $\Rightarrow$ $\vec{A}\vec{x}=\vec{0}$ has an unique solution, which is the trivial solution.

Homogeneous system $\vec{A}\vec{x}=\vec{0}$ has nontrivial solutions if and only if $|\vec{A}|=0$.

If $\vec{A}\vec{x}=\vec{0}$ has one nontrivial solution, $\vec{x}$.

then $\alpha\vec{x}$ must also be nontrivial solution for all $\alpha\in R$ and $\alpha\neq 0$. infinitely many solutions.

Proof. $\vec{A}(\alpha\vec{x}) = \alpha\vec{A}\vec{x} = \alpha\vec{0}=\vec{0}$.

If $\vec{A}\vec{x}=\vec{0}$ has two nontrivial solutions, $\vec{x}_1$ and $\vec{x}_2$, where $\vec{x}_1\neq\alpha\vec{x}_2$ for $\alpha\in R$.

Then $\beta_1\vec{x}_1+\beta_2\vec{x}_2$ must also be nontrivial solution ~~for all non zero~~ ~~$\beta_1, \beta_2 \in R$~~. for all ~~$\beta_1\neq\beta_2$ $\beta_1\beta_2\neq 0$~~ real number β_1, β_2 that are not zero at the same time.

Proof. $\vec{A}(\beta_1\vec{x}_1+\beta_2\vec{x}_2)=\beta_1\vec{A}\vec{x}_1+\beta_2\vec{A}\vec{x}_2=\beta_1\vec{0}+\beta_2\vec{0}=\vec{0}$.

zero rows ~~does~~ do not imply infinitely many solutions !

Ex. $\begin{matrix} 1 & 2 & 1 & \vdots & 1 \\ 2 & 4 & 2 & \vdots & 3 \end{matrix}$ $\quad \left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \\ 1 & 2 & 1 & 1 \end{array}\right) \overset{-1}{\sim} \left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \quad \begin{aligned} x+2y+z &= 1 \\ 2(x+2y+z) &= 3 \end{aligned} \quad \Rightarrow$ no solution.

Ex. $\vec{A}_u \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ k \\ ku \end{pmatrix} \qquad \vec{A}_u = \begin{pmatrix} 1 & 2u-1 & 1-u \\ u-1 & 1 & 3u-1 \\ 0 & u & 2u \end{pmatrix}$

Question: (i) unique solution
(ii) no solution
(iii) infinitely many solutions.

(1). Calculate $\det(\vec{A}_u)$.

$$|\vec{A}_u| = u \cdot (-1)^5 \begin{vmatrix} 1 & 1-u \\ u-1 & 3u-1 \end{vmatrix} + 2u \cdot (-1)^6 \begin{vmatrix} 1 & 2u-1 \\ u-1 & 1 \end{vmatrix}$$

$$= u\cdot(-1)\left(3u-1+(u-1)^2\right) + 2u\left(1 - \underbrace{(u-1)(2u-1)}_{2u^2-3u+1}\right)$$

$$= u(\cancel{1}-3u-u^2-\cancel{1}+2u+\cancel{2}-4u^2+6u-\cancel{2})$$

$$= u(-5u^2+5u)$$

$$= 5u^2(1-u) .$$

$|\vec{A}_u| \neq 0 \iff u \neq 0$ and $u \neq 1$.

when $u \notin \{0, 1\}$, unique solution.

(2). Discuss the cases where $|\vec{A}_u| = 0$.

(2i) $u=0$.

$$\left(\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & k \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{1}{\sim} \left(\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$k=0$. two degrees of freedom. infinitely many solutions.

$k \neq 0$. no solution.

2ii). $u = 1$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & k \\ 0 & 1 & 2 & k \end{array}\right) \begin{array}{c} \\ -1 \\ \leftarrow \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & k \\ 0 & 0 & 0 & 0 \end{array}\right)$$

look at it, no inconsistency, done.

Or. Let $z = t \in \mathbb{R}$.

$$\begin{aligned} x+y &= 1 \\ y+2z &= k \end{aligned} \Rightarrow y = k-2t. \Rightarrow x = 1-y = 1-k+2t. \Rightarrow$$

t could be any real number, so infinitely many solutions.

Conclusion:

- No solution when $u=0$, $k \neq 0$.
- Two degrees of freedom when $u=0, k=0$.
- One degrees of freedom when $u=1$.
- Unique solution when $u \notin \{0,1\}$.