Max / min (& friends ...)

Some basics

If f is a function (that returns $f(x) = y \in \mathbb{R}$)
then a maximum for f [also called: a global maximum]
is an x^* such that

$$f(x^*) \geq f(x), \quad (\text{all } x \neq x^*)$$

It is strict if $f(x^*) > f(x)$, all $x \neq x^*$.

Note: it is the input. The greatest output $f(x^*)$ is called the maximum value.

Minimum, strict minimum: analogous.

x^* is an extremum if it is a maximum or a minimum.

Then: $f(x^*)$ an extreme value.

Max/min cont'd

The concept allows f to be defined on whatever set we may wish, e.g., {bicycle, car, train}

This course: focus on subsets of $\mathbb{R}$ ← (one real variable x)
or $\mathbb{R}^2$ and a little bit on $\mathbb{R}^n$ for general n.
↑ two real var's (x,y)

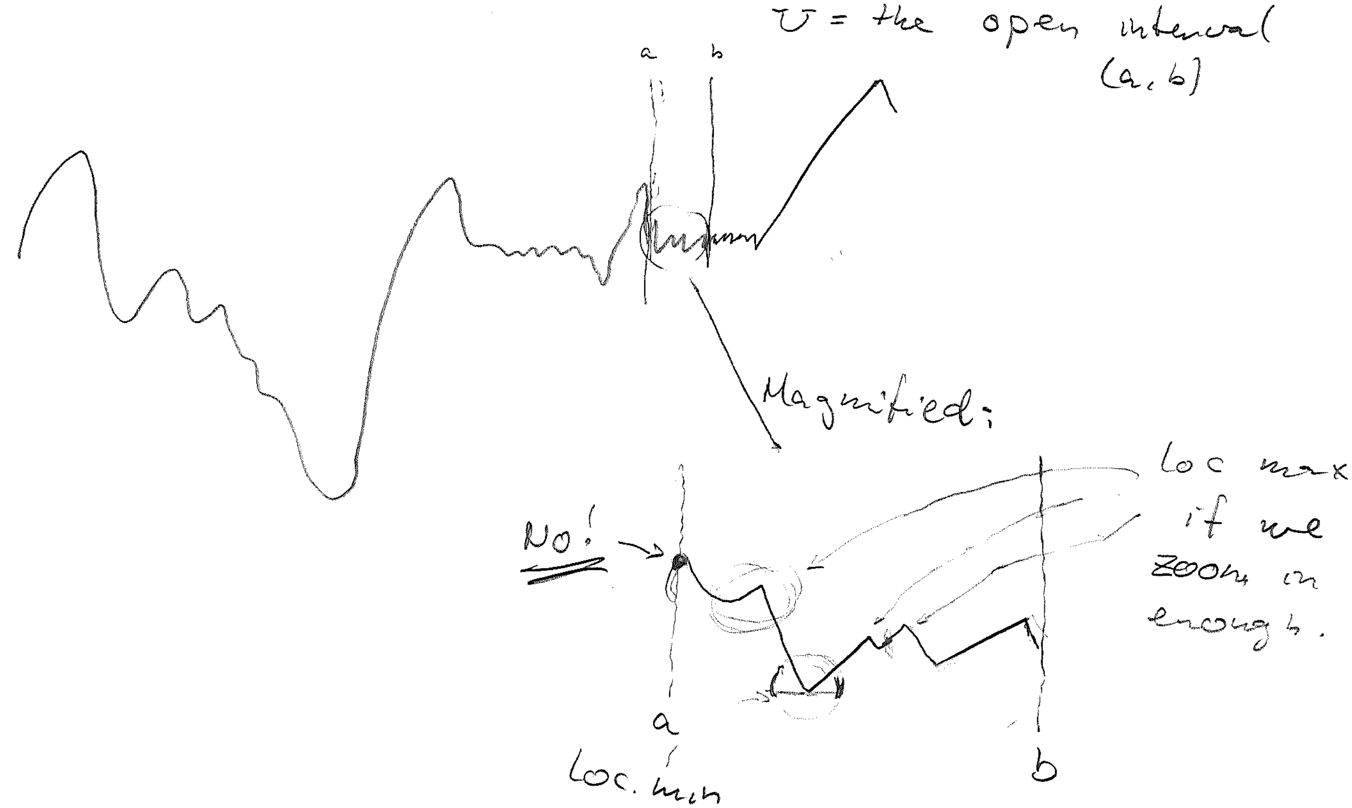
Then we can talk about <u>local</u> max/min:

- Recall that subset U of $\mathbb{R}^n$ is <u>open</u> if it contains none of its boundary points.
- A point P [x^* or (x^*, y^*) or $(x_1^*, \ldots, x_n^*)$] is a <u>local maximum</u> for f if:
  - → there exists some open U centered at P such that
  - → we can zoom in on U and <u>discard</u> everything else (let $g = f$ on U, <u>undefined otherwise</u>)
  - → and then get a maximum.

Strict loc. max, loc min, strict loc. min, loc. extrema ...

Maximum: local

Like this ($n=1$):



Why the "open" set?

See the "No!" point. Need "proper inside".

(Local) max/min: questions

Q1: What could they possibly look like?

Q2: How to find possible candidates for max/min?

Q3: How to decide once one is found, whether it is a local max/min – and if so, a global as well?

But.... waitaminute ....

Q0: Can we even know that one exists?

$f(x) = x$, defined for all x? No!

$f(x) = x$, defined for $x \in [-5, 2]$? Yes!

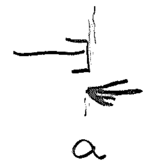
$\leadsto$ The extreme value theorem.

(We will get to it.)

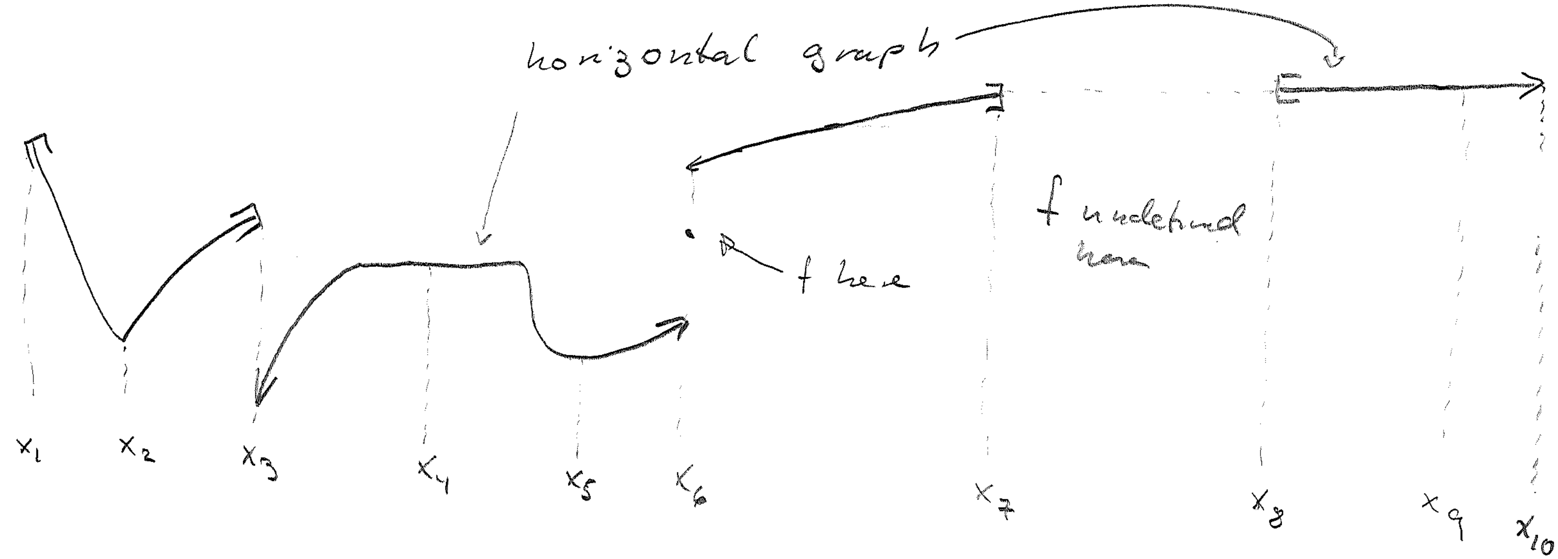
A1: What they could look like

- Discontinuity points — Not important in Math 2
- Nondifferentiability points — You must know that $|x|$ has a minimum at $x=0$
- Stationary points (where $f'(x)=0$; $\frac{\partial f}{\partial x_i}(x_1 \ldots x_n) = 0 \quad \forall i$) — Essential!!!
- Boundary points
  $\hookrightarrow$ endpoints when $n=1$ — Quite a lot of attention, especially for $n>1$.

$n=1$, illustration



means: $f(x^*) = \lim_{x \to a^-} f(x)$



Which ones are local max? Local min?

Global max? Global min?

Strictly so?

A3: how to decide

* Left endpoint: $\quad a \quad$ f defined at a.

$f' > 0$ at a $\Rightarrow$ loc. min

$f' < 0$ at a $\Rightarrow$ loc. max

$f' = 0$ at a .... ?

Depends on what happens just to the right of a.

* Right endpoint: switch sign! (why?)

* Stationary points:

→ first-derivative test

→ second-derivative test.

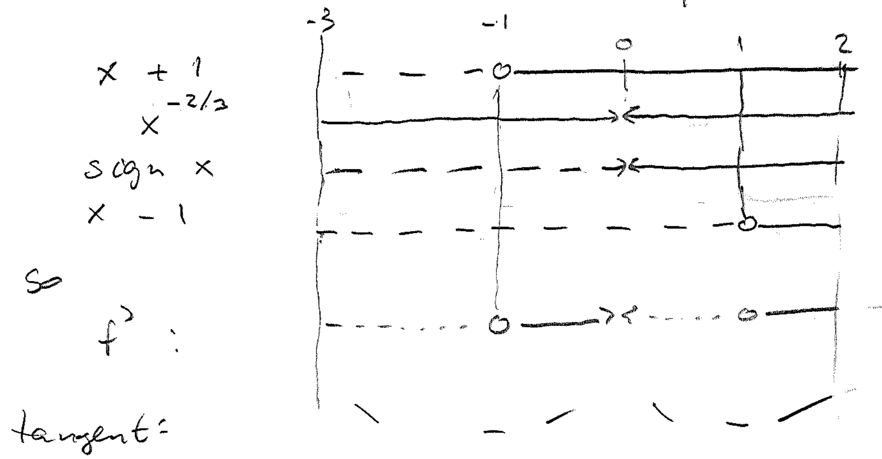
* Stationary points. The first-derivative test

Ex.: If $f(x) = |x|^{7/3} - 7|x|^{1/3}$, $x \in [-3, 2]$

$$f'(x) = \frac{7}{3}(x-1)(x+1)\cdot x^{-2/3} \cdot \operatorname{sign} x$$

sign $x = \frac{d}{dx}|x|$, $x \neq 0$.

as a product.



$x=-1$: loc. min. $x = 1$: loc. min.

$x = -3$, $x = 0$ (a cusp!) and $x=2$: loc max.

Q: Global ...?

* The second-derivative test: later!

$n > 1$:

* Boundary points:
  Not so easy anymore!
  - An interval has two endpoints.
  - A set like the first quadrant $\{(x,y);\ x \geq 0,\ y \geq 0\}$ has infinitely many!

This is why we have a theory for constrained max/min!

→ Lagrange's method

→ Kuhn–Tucker cond's.

* Stationary points:
  First-derivative test ... ?
  Second-derivative test!

* Second derivatives,

concavity / convexity

max / min :

Time to take notes!

$f'' > 0$?

$f'' \geq 0$?

f''_{xx} and f''_{yy} and

$f''_{xx} f''_{yy} - (f''_{xy})^2$

> 0 ?

≥ 0 ?