University of Oslo / Department of Economics **English only***

ECON3120/4120 Mathematics 2

Compulsory term paper, spring term 2018.

There are 2 pages of problems to be solved, not counting this page.

Justifying answers:

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Minimum requirements to pass this assignment:

- You will pass if *each of problems 1, 2, and 3* is scored as good enough to pass, if it were judged to be one exam stand-alone.
  (The commonly applied pass mark in mathematics is forty percent, and this course by default uses uniform weighting over letter-enumerated items.)
- If you fail one of the three problems despite a decent attempt at it, we may still let you pass upon judging the overall quality of the paper. In particular, we shall take into account what parts are acknowledged as demanding.

The paper does *not* count towards your final course grade!

- Passing the term paper is required in order to sit in on the exam (see the Department's rules). Other than that, it does not in any way count towards your grade, and the exam grading committee will not see your term paper.

If you have questions concerning ...

- Questions concerning the problem set (the mathematics): ask the teachers.
- All other questions: `mailto:post@econ.uio.no`
- And, keep an eye on the course website – in particular the Messages section – in case of any clarifications or other information.

*Answers can be given in English, Norwegian, Swedish or Danish, as per the regulations for the exam.

Problem 1

(a) Let $h(x) = x^e \cdot [e^{px} - \frac{1}{\ln(1+x)}]$ and $H(x) = \frac{\ln(1+h(x))}{h(x)}$. Consider for each $p \neq 0$ the limits

$$\lim_{x \to 0^+} H(x) \qquad \text{and} \qquad \lim_{x \to +\infty} H(x).$$

For each $p \neq 0$:[†] what does l'Hôpital's rule tell you about each of these limits?

(b) For p in a certain range, the function $h(x)$ of part (a) will have a global extremum $x^* > 0$. The extreme value $V = h(x^*)$ depends on p. Find an expression for $V'(p)$.

(c) Let $u(x) = x - x^{-x}$ and $v(x) = \log_{\sqrt{x}}(1 - u(x))$ (the log has $\sqrt{x}$ as base!). Find $v'(x)$.

In the following, let $w(s,t)$ be a given function that is everywhere twice continuously differentiable, and define $f(x,y) = w(x^2, xy^2) + 2018(x^3+1)$.

(d) Show that the origin is a stationary point for f, and check whether the second-derivative test can classify this point.
(The answer could depend on the behaviour of the partial derivatives of w.)

From now on, let $r > 0$ be a constant and $w(s,t) = t \cdot (e^{rs} - rs - 1)$.

(e) In this part, you shall show *by contradiction* that the stationary point $(0,0)$ for f is a saddle point, *without* invoking second derivatives. You can proceed the following way:
- Since it is given in (d) that we have a stationary point: assume for contradiction that it is a local minimum. Show that somewhere arbitrarily close to $(0,0)$, we have points where $f(x,y) - f(0,0)$ is < 0.
- Then do the analogous argument to disprove local maximum.

Hint: compare $f(-x,0)$ with $f(x,0)$.

(f) Consider the problem

$$\max\ f(x,y) \qquad \text{subject to} \qquad x^2 + y^2 \leq r^2 \qquad x \geq 0 \qquad \text{and} \qquad y \geq 0 \tag{P}$$

- State the Kuhn–Tucker conditions, and explain why they are satisfied at $(0,0)$.
- Why can we tell that problem (P) has a solution?

Problem 2 Compendium problem 28.[‡]

Note the partial answers given on page 36; for those questions, you shall show that the answers on p. 36 are correct. For the others: solve completely.
You are allowed to use the page 36 answers given for *letter items earlier in the alphabet* than the one you apply them to; e.g., in your answer to "(c)" you can freely use what page 36 states on parts "(a)" and "(b)", no matter whether you completed those two parts.

[†] when it says "each $p \neq 0$", that means you must also check $p < 0$. Be warned that you can *not* expect this reminder on the exam!

[‡] Some parts could be demanding, and you may want to do the seminar-assigned problem 32 first. *Hints:* take note what part (b) does (not) ask for; in (c), maybe sketch what the graph must look like for $x \in (0, a/2)$; likely f'' is not the easiest tool to answer (d); in (f), $0 < x_0 < a/2$ which $\to 0$ as $a \to 0^+$.

Problem 3 Let r, s, t, u, v be positive constants. Define the matrices

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{pmatrix} s & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & s \end{pmatrix}$$

(a) Calculate $\mathbf{U}^2$ and $(r\mathbf{L} + \mathbf{D} + u\mathbf{U})\mathbf{L}$.

(b) Put $s = t = 1$. In this part, you shall use Gaussian elimination[§] to solve for the inverse of $(\mathbf{D} + u\mathbf{U})$ or show that it does not exist, for every $u > 0$. That is, you shall solve the equation system $\mathbf{AX} = \mathbf{I}$, where $\mathbf{A} = \mathbf{D} + u\mathbf{U}$. Note that the unknown $\mathbf{X}$ and the identity matrix $\mathbf{I}$ are both 5×5.

- For full score:
  Write down the augmented coefficient matrix $(\mathbf{A} \vdots \mathbf{I})$ and perform Gaussian elimination on this until you have found the solution or shown that no solution exists.
- For up to 2/3 score (usually corresponding to a near-middle-of-the-road "C"):
  Solve the three equation systems $\mathbf{Ax} = \big(1,\ 0,\ 0,\ 0,\ 0\big)'$, $\mathbf{Ay} = \big(0,\ 1,\ 0,\ 0,\ 0\big)'$ and $\mathbf{Az} = \big(0,\ 0,\ 1,\ 0,\ 0\big)'$ and explain then where/how $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ enter the inverse and how you would solve the rest.

(c) Let $\mathbf{0}_{n\times 1}$ denote the n-column vector with all elements zero, and let $\mathbf{w}$ be the column vector $\mathbf{w} = (1, 2, \ldots, n)'$ (order $n \times 1$), where $n > 1$.
Assume that a matrix $\mathbf{M}$ is such that $\mathbf{Mw} = \mathbf{0}_{n\times 1}$.
For each statement (I)–(V) below, decide whether it is always true (i.e. must be true for all $\mathbf{M}$ as specified above) cannot be true (not for any such $\mathbf{M}$), or neither.[¶]

  (I) $\mathbf{M} = \mathbf{0}$.

  (II) The equation system $\mathbf{Mx} = \mathbf{0}$ has precisely two solutions: $\mathbf{x} = \mathbf{w}$ or $\mathbf{x} = \mathbf{0}$.

  (III) $\mathbf{M}$ is square.

  (IV) $\mathbf{M}$ has an inverse.

  (V) $\mathbf{M}$ is symmetric.

For some of these claims, a look at the case $n = 2$ could give a hint.

(d) [‖]Let $\mathbf{K}$ and $\mathbf{L}$ be matrices satisfying $\mathbf{K}^{-1}\mathbf{L} = \mathbf{L}\mathbf{K}^{-1}$. Show that $\mathbf{KL} = \mathbf{LK}$.

[§]Note that the question tells you to use that particular method. You will not get credit for using the formula for the inverse!

[¶]"Neither" means, e.g. for (V): there is an example where $\mathbf{Mw} = \mathbf{0}_{n\times 1}$ and $\mathbf{M}$ is symmetric, and an example where $\mathbf{Mw} = \mathbf{0}_{n\times 1}$ and $\mathbf{M}$ not symmetric.

[‖]This is likely the problem that requires the most recent curriculum prior to the deadline. In case of delays in the lecture plan, we might choose to give you an option to solve a different problem instead – in which case you *will* have the option to answer the problem as given above.