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ECON 4130

Supplementary Exercises II

Exercise 5

a) Let $Z \sim N(0,1)$ with cdf, $\Phi(x) = P(Z \le x)$. Put $Y = a + bZ$ where *a* and *b* are constants. Show that the cdf of *Y* is

$$F_Y(y) = P(Y \le y) = \begin{cases} \Phi\left(\dfrac{y-a}{b}\right) & \text{if } b > 0 \\ 1 - \Phi\left(\dfrac{y-a}{b}\right) & \text{if } b < 0 \end{cases}$$

[**Hint:** Manipulate $P(Y \le y)$.] This implies that $Y \sim N(a, b^2)$. Why?

b) Let Z_n is a r.v. with cdf, $F_n(x) = P(Z_n \le x)$ for $n = 1, 2, \ldots$, and suppose that $Z_n \underset{n\to\infty}{\overset{D}{\to}} Z \sim N(0,1)$. Show by using the definition of convergence in distribution directly, that $Y_n = a + bZ_n \underset{n\to\infty}{\overset{D}{\to}} Y \sim N(a, b^2)$ [**Hint:** Treat the cases, $b > 0, b < 0$ separately, and look at $P(Y_n \le y)$.]

c) Let A_n, B_n be r.v.'s such that, $A_n \underset{n\to\infty}{\overset{P}{\to}} a,$ and $B_n \underset{n\to\infty}{\overset{P}{\to}} b$. Show that $Y_n = A_n + B_n Z_n \underset{n\to\infty}{\overset{D}{\to}} Y \sim N(a, b^2)$. [**Hint:** Use Slutzky's lemma.]

d) Let θ be an unknown parameter in a model and $\hat{\theta}$ an estimator based on *n* observations. Suppose we have proved that $\sqrt{n}(\hat{\theta} - \theta) \underset{n\to\infty}{\overset{D}{\to}} X \sim N(0, b^2)$ where *b* is some constant. Show that this implies that $\hat{\theta}$ must be a consistent estimator for θ.

[**Hint:** Set $X_n = \sqrt{n}(\hat{\theta} - \theta)$, solve with respect to $\hat{\theta}$, and use Slutzky. Also properties (3) and (5) in "Lecture notes II to Rice chap. 5" may be relevant.]

Exercise 6 (the δ-method for determining asymptotic normality of estimators).

Let θ be an unknown parameter in a model and $\hat{\theta}$ an estimator based on *n* observations. Suppose we have proved that $\sqrt{n}(\hat{\theta}-\theta)\underset{n\to\infty}{\overset{D}{\to}} X \sim N(0,b^2)$ where *b* is some constant. Suppose that what we are really interested in is another transformed parameter, $\gamma = g(\theta)$. We assume that $g(x)$ is a continuously differentiable function everywhere where θ and $\hat{\theta}$ may take their values. It is natural to estimate γ by $\hat{\gamma} = g(\hat{\theta})$ (in fact, both the moment method (MM) and the maximum likelihood method (ML) recommend us to do so if $\hat{\theta}$ is a MM- or ML- estimator respectively). To measure the uncertainty of this estimation, we need a $1-\alpha$ confidence interval for γ, at least approximately. This we can achieve since it can be proven that, under these conditions,

(1) $$\sqrt{n}(\hat{\gamma}-\gamma)\underset{n\to\infty}{\overset{D}{\to}} Y \sim N(0,b^2[g'(\theta)]^2)$$

a) Proof of (1): The proof follows simply from the first order Taylor expansion of $\hat{\gamma} = g(\hat{\theta})$ around θ (see (A3) in appendix 1 in "Lecture notes II to Rice chap. 5" (referred to as LN2).

(2) $$\hat{\gamma} = g(\hat{\theta}) = g(\theta) + g'(\theta)\cdot(\hat{\theta}-\theta) + R_n,$$ where R_n is an error term.

Now, it can be shown that $\sqrt{n}R_n \underset{n\to\infty}{\overset{P}{\to}} 0$ [for those interested a proof can be found in the appendix to this exercise]. Use this and Slutzky's lemma to prove (1).
[**Note:** This way of deriving a limit distribution by a first order Taylor expansion, is often referred to in the literature as the so called "delta-method".]

b) An approximate $1-\alpha$ confidence interval for γ: Usually the constant *b* is also unknown. We assume that we have an estimator, $\hat{b}$, that is consistent for *b*. We then get from (1)

(3) $$P\left(\hat{\gamma} - z_{\frac{\alpha}{2}}\frac{\sqrt{\hat{b}^2[g'(\hat{\theta})]^2}}{\sqrt{n}} \le \gamma \le \hat{\gamma} + z_{\frac{\alpha}{2}}\frac{\sqrt{\hat{b}^2[g'(\hat{\theta})]^2}}{\sqrt{n}}\right) \approx 1-\alpha \quad \text{for large } n$$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$-point (i.e. the $1-\frac{\alpha}{2}$ percentile) in $N(0,1)$. Justify (3) by using Slutzky's lemma along the lines in example 5 in LN2.

Appendix to exercise 6

Proof that $\sqrt{n}R_n \underset{n\to\infty}{\overset{P}{\to}} 0$ in (2):

Taylor-expanding $\hat{\gamma} = g(\hat{\theta})$ around θ with one term plus error gives (see (A3) in appendix 1 in LN2)

(4) $\quad \hat{\gamma} = g(\hat{\theta}) = g(\theta) + g'(c_n)(\hat{\theta} - \theta)$

where c_n lies somewhere between θ and $\hat{\theta}$. Write (4) as

$$\hat{\gamma} = g(\hat{\theta}) = g(\theta) + g'(\theta)(\hat{\theta} - \theta) + (g'(c_n) - g'(\theta))(\hat{\theta} - \theta) = g(\theta) + g'(\theta)(\hat{\theta} - \theta) + R_n$$

where $R_n = (g'(c_n) - g'(\theta))(\hat{\theta} - \theta)$. Now $\sqrt{n}R_n = A_n U_n$ where $A_n = g'(c_n) - g'(\theta)$ and $U_n = \sqrt{n}(\hat{\theta} - \theta)$. If we can prove that $A_n \overset{P}{\to} 0$, it follows from Slutzky (since U_n converges in distribution) and properties (3) and (5) in LN2, that $\sqrt{n}R_n \underset{n\to\infty}{\overset{P}{\to}} 0$ is true. First observe that $c_n \underset{n\to\infty}{\overset{P}{\to}} \theta$. This follows since c_n is between θ and $\hat{\theta}$, which implies that $|c_n - \theta| \le |\hat{\theta} - \theta|$. Then, if $\varepsilon > 0$ is arbitrarily small the following events satisfy $(|c_n - \theta| > \varepsilon) \Rightarrow (|\hat{\theta} - \theta| > \varepsilon)$, from which follows (note that $\hat{\theta}$ is consistent from exercise 5d) that $P(|c_n - \theta| > \varepsilon) \le P(|\hat{\theta} - \theta| > \varepsilon) \underset{n\to\infty}{\to} 0$, and, therefore, $P(|c_n - \theta| > \varepsilon) \underset{n\to\infty}{\to} 0$.

Now, $g'(x)$ is continuous, and therefore $A_n = g'(c_n) - g'(\theta) \underset{n\to\infty}{\overset{P}{\to}} 0$ (see theorem 1 in LN2).
Q.E.D.