HG
Oct. 05

ECON 4130

Supplementary Exercises 5 and 6

(Applications of Slutsky's lemma)

Exercise 5

a) Let $\hat{\theta}_n$ be an estimator for an unknown parameter θ. Suppose we know that $\sqrt{n}\left(\hat{\theta}_n - \theta\right) \underset{n\to\infty}{\overset{D}{\to}} Z \sim N(0, b^2)$ where $b > 0$ is a constant. Explain why this is the same as saying that $\sqrt{n}\frac{\hat{\theta}_n - \theta}{b} \underset{n\to\infty}{\overset{D}{\to}} \frac{Z}{b} \sim N(0,1)$. [**Hint:** Use the definition of convergence in distribution directly, or, alternatively, use Slutsky's lemma combined with property (3) in "Lecture Notes to Rice chap. 5".]

b) Let θ be an unknown parameter in a model and $\hat{\theta}_n$ an estimator based on *n* observations. Suppose we have proved that $\sqrt{n}(\hat{\theta}_n - \theta) \underset{n\to\infty}{\overset{D}{\to}} X \sim N(0, b^2)$ where $b > 0$ is some constant. Show that this implies that $\hat{\theta}_n$ must be a consistent estimator for θ.

[**Hint:** Set $X_n = \sqrt{n}(\hat{\theta}_n - \theta)$, solve with respect to $\hat{\theta}_n$, and use Slutsky. Also properties (3) and (5) in "Lecture Notes to Rice chap. 5" may be relevant.]

Exercise 6 (the δ-method for determining asymptotic normality of estimators).

Let θ be an unknown parameter in a model and $\hat{\theta}$ an estimator based on *n* observations. Suppose we have proved that $\sqrt{n}(\hat{\theta} - \theta) \underset{n\to\infty}{\overset{D}{\to}} X \sim N(0, b^2)$ where *b* is some constant. Suppose that what we are really interested in is another transformed parameter, $\gamma = g(\theta)$. We assume that $g(x)$ is a continuously differentiable function everywhere where θ and $\hat{\theta}$ may take their values. It is natural to estimate γ by $\hat{\gamma} = g(\hat{\theta})$ (in fact, both the moment method (MM) and the maximum likelihood method (ML) recommend us to do so if $\hat{\theta}$ is a MM- or ML- estimator respectively). To measure the uncertainty of this estimation, we need a $1-\alpha$ confidence interval for γ, at least approximately. This we can achieve by the following lemma (combined with Slutsky's lemma):

Lemma *Let* $\sqrt{n}(\hat{\theta}-\theta)\underset{n\to\infty}{\overset{D}{\to}} X \sim N(0,b^2)$, *and let* $g(x)$ *be a continuously differentiable function everywhere where* θ *and* $\hat{\theta}$ *may take their values. Then also*

(1) $$\sqrt{n}\left(g(\hat{\theta})-g(\theta)\right)\underset{n\to\infty}{\overset{D}{\to}} Y \sim N(0,b^2[g'(\theta)]^2)$$

[With other words: $\sqrt{n}(\hat{\gamma}-\gamma)\underset{n\to\infty}{\overset{D}{\to}} Y \sim N(0,b^2[g'(\theta)]^2)$ where $\hat{\gamma}=g(\hat{\theta})$ and $\gamma=g(\theta)$]

a) **Proof of (1):** The proof follows simply from the first order Taylor expansion of $\hat{\gamma}=g(\hat{\theta})$ around θ (see (A3) in appendix 1 in "Lecture Notes to Rice chap. 5" (referred to as LN5).

(2) $$\hat{\gamma}=g(\hat{\theta})=g(\theta)+g'(\theta)\cdot(\hat{\theta}-\theta)+R_n,$$ where R_n is an error term.

Now, it can be shown that $\sqrt{n}R_n\underset{n\to\infty}{\overset{P}{\to}} 0$ [for those interested a proof can be found in the appendix to this exercise]. *Use this and Slutsky's lemma to prove (1).*

[**Note:** This way of deriving a limit distribution by a first order Taylor expansion, is often referred to in the literature as the so called "delta-method".]

b) **An approximate $1-\alpha$ confidence interval for γ:** Usually the constant *b* is also unknown. We assume that we have an estimator, $\hat{b}$, that is consistent for *b*. We then get from (1)

(3) $$P\left(\hat{\gamma}-z_{\frac{\alpha}{2}}\frac{\sqrt{\hat{b}^2[g'(\hat{\theta})]^2}}{\sqrt{n}}\le\gamma\le\hat{\gamma}+z_{\frac{\alpha}{2}}\frac{\sqrt{\hat{b}^2[g'(\hat{\theta})]^2}}{\sqrt{n}}\right)\approx 1-\alpha \quad \text{for large } n$$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$-point (i.e. the $1-\frac{\alpha}{2}$ percentile) in $N(0,1)$. *Justify (3) by using Slutsky's lemma along the lines in example 5 in LN5.*

c) Suppose $X_1, X_2, \ldots, X_n$ ~ *iid*, with common *pmf*:
$f(x;p) = P(X_i = x) = p(1-p)^{x-1}$ for $x = 1, 2, 3, \ldots$.

Put $\theta = \dfrac{1}{p} = \mathrm{E}(X_i)$. Explain why the central limit theorem (CLT) implies that
$\sqrt{n}\left(\bar{X} - \theta\right) \underset{n\to\infty}{\overset{D}{\to}} Z \sim N\left(0,\ \dfrac{1-p}{p^2}\right)$. Use this and the lemma in **a.** to show that $\hat{p} = 1/\bar{X}$ (i.e., the *mle* and *mme* estimator according to Written paper II) is approximately normally distributed in the sense

$$\sqrt{n}\left(\hat{p} - p\right) \underset{n\to\infty}{\overset{D}{\to}} Y \sim N(0,\ p^2(1-p))$$

(which can be interpreted as $\hat{p} \overset{Approx.}{\sim} N\left(p,\ \dfrac{p^2(1-p)}{n}\right)$ for given *n*. Note also that the result can be derived by the asymptotic theory of mle – estimators, e.g., see “written paper II”.)

Appendix to exercise 6

Proof that $\sqrt{n}R_n \underset{n\to\infty}{\overset{P}{\to}} 0$ in (2):

Taylor-expanding $\hat{\gamma} = g(\hat{\theta})$ around θ with one term plus error gives (see (A3) in appendix 1 in LN5)

(4) $\hat{\gamma} = g(\hat{\theta}) = g(\theta) + g'(c_n)(\hat{\theta} - \theta)$

where c_n lies somewhere between θ and $\hat{\theta}$. Write (4) as

$$\hat{\gamma} = g(\hat{\theta}) = g(\theta) + g'(\theta)(\hat{\theta} - \theta) + (g'(c_n) - g'(\theta))(\hat{\theta} - \theta) = g(\theta) + g'(\theta)(\hat{\theta} - \theta) + R_n$$

where $R_n = (g'(c_n) - g'(\theta))(\hat{\theta} - \theta)$. Now $\sqrt{n}R_n = A_n U_n$ where $A_n = g'(c_n) - g'(\theta)$ and $U_n = \sqrt{n}(\hat{\theta} - \theta)$. If we can prove that $A_n \overset{P}{\to} 0$, it follows from Slutsky (since U_n converges in distribution) and properties (3) and (5) in LN5, that $\sqrt{n}R_n \underset{n\to\infty}{\overset{P}{\to}} 0$ is true. First observe that $c_n \underset{n\to\infty}{\overset{P}{\to}} \theta$. This follows since c_n is between θ and $\hat{\theta}$, which implies that $|c_n - \theta| \leq |\hat{\theta} - \theta|$. Then, if $\varepsilon > 0$ is arbitrarily small the following events satisfy

$(|c_n - \theta| > \varepsilon) \Rightarrow (|\hat{\theta} - \theta| > \varepsilon)$, from which follows (note that $\hat{\theta}$ is consistent from exercise 5b) that $P(|c_n - \theta| > \varepsilon) \leq P(|\hat{\theta} - \theta| > \varepsilon) \underset{n\to\infty}{\to} 0$, and, therefore,

$P(|c_n - \theta| > \varepsilon) \underset{n\to\infty}{\to} 0$. Hence $c_n \underset{n\to\infty}{\overset{P}{\to}} \theta$.

Now, $g'(x)$ is continuous, and therefore $A_n = g'(c_n) - g'(\theta) \underset{n\to\infty}{\overset{P}{\to}} 0$ (see theorem 1 in LN5). Q.E.D.