HG Nov 16

Answers to three exercises in the lecture notes to Rice chapter 8

Exercise 1 (page 2)

(i) If $X \sim \text{bin}(n, p)$, we have according to the general theory for the binomial distr. that, for large *n*, $X \overset{approx.}{\sim} N\left(E(X), \text{var}(X)\right) = N(np, np(1-p))$, or

$$\frac{X - np}{\sqrt{np(1-p)}} \overset{approx.}{\sim} N(0,1)$$

But

$$\frac{X - np}{\sqrt{np(1-p)}} = \frac{n(X/n - p)}{\sqrt{n}\sqrt{p(1-p)}} = \sqrt{n}\,\frac{\hat{p} - p}{\sqrt{p(1-p)}} \overset{approx.}{\sim} N(0,1)$$

Hence,

$b = \sqrt{p(1-p)}$ and $\hat{b} = \sqrt{\hat{p}(1-\hat{p})}$

where $\hat{p} = X/n$ is consistent, which implies by continuity that also $\hat{b}$ is consistent for *b*.

(ii) According to theory, if $X \sim \text{poisson}(t\lambda)$ (writing *t* for *n*), then $X \overset{approx.}{\sim} N\left(E(X), \text{var}(X)\right) = N(\lambda t, \lambda t)$ for large t (i.e., such that $t\lambda \geq 10$, say). As in (i) we get

$$\frac{X - t\lambda}{\sqrt{t\lambda}} = \frac{t(X/n - \lambda)}{\sqrt{t}\sqrt{\lambda}} = \sqrt{t}\,\frac{\hat{\lambda} - \lambda}{\sqrt{\lambda}} \overset{approx.}{\sim} N(0,1)$$

Hence $b = \sqrt{\lambda}$ and $\hat{b} = \sqrt{\hat{\lambda}} = \sqrt{\frac{X}{t}}$ is consistent for *b*.

(iii) According to the central limit theorem, for large *n*,

$$\bar{X} \overset{approx.}{\sim} N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(E(\bar{X}), \text{var}(\bar{X})\right).$$

Hence $b = \sigma$, and, for example, $\hat{b} = S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$ is consistent for *b*.

Exercise 2 (page 11)

(a)

From $X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$ and $X' = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$, we get

$$X'X = \begin{pmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{pmatrix}$$

from which the determinant follows directly.

(b)

The inverse of a 2×2- matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is in general given by

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ (check by multiplying A and A^{-1}**)**

where $D = ad - bc$ is the determinant of *A*, which must be different from 0 for the inverse to exist.

Hence, from (a):

$$(X'X)^{-1} = \frac{1}{n\sum_i (x_i - \bar{x})^2} \cdot \begin{pmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{pmatrix}$$

Since $\text{cov}(\hat{\beta}) = \sigma^2 (X'X)^{-1}$, we find the variances on the main diagonal, giving the expressions in the exercise.

(c) From the matrices in (a) and (b) we get, writing $D = n\sum_i (x_i - \bar{x})^2$ for the determinant,

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (X'X)^{-1} X'Y = \frac{1}{D} \cdot \begin{pmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{pmatrix} \cdot \begin{pmatrix} \sum_i Y_i \\ \sum_i x_i Y_i \end{pmatrix}$$

We now find $\hat{\beta}_1$ as the second element in the vector $\hat{\beta}$:

$$\hat{\beta}_1 = \frac{1}{D}\cdot\left(-\sum_i x_i \sum_i Y_i + n\sum_i x_i Y_i\right) = \frac{1}{D}\cdot\left(n\sum_i x_i Y_i - n^2\overline{x}\overline{Y}\right) =$$

$$= \frac{n\left(\sum_i x_i Y_i - n\overline{x}\overline{Y}\right)}{n\sum_i (x_i-\overline{x})^2} = \frac{\sum_i (x_i-\overline{x})(Y_i-\overline{Y})}{\sum_i (x_i-\overline{x})^2} = \frac{S_{xY}}{S_x^2}$$

Exercise 3 (page 12)

Since the determinant of the covariance matrix $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ is $D = \sigma_{11}\sigma_{22}(1-\rho^2)$, we get the inverse as in exercise 2b

$$\Sigma^{-1} = \frac{1}{D}\begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$

and

$$(x-\mu)'\Sigma^{-1}(x-\mu) = \frac{1}{D}(x_1-\mu_1,\ x_2-\mu_2)\cdot\begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}\cdot(x-\mu) =$$

$$= \frac{1}{D}\left[(x_1-\mu_1)\sigma_{22} - (x_2-\mu_2)\sigma_{12},\quad -(x_1-\mu_1)\sigma_{12} + (x_2-\mu_2)\sigma_{11}\right]\cdot\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix} =$$

$$= \frac{1}{D}\left[(x_1-\mu_1)^2\sigma_{22} - (x_2-\mu_2)(x_1-\mu_1)\sigma_{12} - (x_1-\mu_1)(x_2-\mu_2)\sigma_{12} + (x_2-\mu_2)^2\sigma_{11}\right] =$$

$$= \frac{1}{\sigma_{11}\sigma_{22}(1-\rho^2)}\left[(x_1-\mu_1)^2\sigma_{22} - 2(x_1-\mu_1)(x_2-\mu_2)\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\rho + (x_2-\mu_2)^2\sigma_{11}\right] =$$

$$= \frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right) + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2\right]$$

Multiplying this by $-\frac{1}{2}$, we see that the exponent in (12) reduces exactly to the exponent in Example F in Rice section 3.3. We also see that the expression in the denominator in the pdf in example F (Rice) is the same as in (12) from the expression of the determinant *D*.