

ECON 4130 H16

Extra exercises for no-seminar week 44

(Solutions will be put on the net on Thursday week 44)

Exercise 1

This exercise is based on the Exam 2004H - "utsatt prøve", slightly extended and adapted to fit the present curriculum.

The random variable (rv.), Y , has a log-normal distribution with parameters, μ and σ^2 , if the density function (pdf) is given by

$$f(y) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma}} \cdot \frac{1}{y} \cdot e^{-\frac{1}{2\sigma^2}[\ln(y)-\mu]^2} & \text{for } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

This is a right skewed distribution with a pdf somewhat similar to the pdf of a gamma distribution. It is sometimes used to model income distributions.

A. Show that, if Y is log-normal (μ, σ^2) then $X = \ln(Y)$ is normally distributed with expectation, μ and variance, σ^2 (i.e., $N(\mu, \sigma^2)$).

B. Explain how the moment generating function (mgf) for X , can be utilized to show that

$$E(Y^k) = e^{k\mu + k^2 \frac{\sigma^2}{2}} \quad \text{for } k = 1, 2, 3, \dots$$

C. The *variation coefficient* of a non-negative rv., Z , denoted by $VC(Z)$, is defined as

$$VC(Z) = \frac{\sqrt{\text{Var}(Z)}}{E(Z)}$$

The variation coefficient is a measure of variation. If Z is the income of a person randomly chosen from a population of income earners, $VC(Z)$ is sometimes taken as a measure of income inequality for the population in question.

- (i) Show that the VC is invariant for scale transformations, i.e., show that $VC(cZ) = VC(Z)$ for any constant $c > 0$.
- (ii) Let Y be log-normal (μ, σ^2) . Show that the variation coefficient, that we will denote by γ , is $\gamma = VC(Y) = \sqrt{e^{\sigma^2} - 1}$
- (iii) Let Y be gamma distributed, (α, λ) , where α is the shape parameter and λ the scale parameter. Show that the variation coefficient is equal to $1/\sqrt{\alpha}$ and, hence, independent of the scale λ .

D. Let Z_1, Z_2, \dots, Z_n be *iid* and non-negative rv's with expectation, $E(Z_i) = \eta$, and variance, $\text{Var}(Z_i) = \tau^2$. Otherwise we don't know anything about the common distribution of the Z_i 's. Propose a consistent estimator for the VC in this case, and explain why it is consistent.

E. Let Y_1, Y_2, \dots, Y_n be *iid* and log-normally distributed (μ, σ^2) . Show that the maximum likelihood estimators (MLE's) for μ and σ^2 are given by

$$\hat{\mu} = \overline{\ln(Y)} = \frac{1}{n} \sum_{i=1}^n \ln(Y_i) \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n [\ln(Y_i) - \overline{\ln(Y)}]^2 \quad \text{respectively. [Hint: Compare}$$

the log likelihood function with the log likelihood for a normal sample, i.e., study example B in Rice section 8.5]

What is the MLE for the variation coefficient, $\gamma = VC(Y_i)$?

F. Derive the moment estimators (MME's) for μ, σ^2 , and γ , based on Y_1, Y_2, \dots, Y_n in question **E**.

G. We have a sample of $n = 121$ yearly incomes drawn from a population of women (Norway 1998) that is relatively homogenous with regard to the time spent at paid work. Let Y_i denote the income of woman i in the sample. As before we assume that Y_1, Y_2, \dots, Y_n is *iid* and log-normally distributed (μ, σ^2) . Calculate both the MLE- and MME estimates of the population VC, γ , based on the summary data in the table

<i>Statistic</i>	<i>Data</i>
n (sample size)	121
$\frac{1}{n} \sum_{i=1}^n Y_i$ (NOK)	202 799
$\frac{1}{n} \sum_{i=1}^n Y_i^2$	46 597 545 146
$\frac{1}{n} \sum_{i=1}^n \ln(Y_i)$	12.15916
$\frac{1}{n} \sum_{i=1}^n [\ln(Y_i)]^2$	147.96481

H. It can be shown that the MLE, $\hat{\sigma}^2$, is asymptotically normally distributed in the sense $\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow[n \rightarrow \infty]{D} N(0, 2\sigma^4)$ (You do not need to show this here.) Use this to develop an asymptotic 95% confidence interval (CI) for γ based on a corresponding CI for σ^2 . Calculate the interval.

I. A well known fact is that if X_1, X_2, \dots, X_n are *iid* and normally distributed,

$X_i \sim N(\mu, \sigma^2)$, then $\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2}$ is (exactly) χ^2 distributed with $n-1$ degrees of freedom. Use this to find an exact 95% CI for σ^2 , and, from this, an exact 95% CI for γ . Calculate the interval and compare with the approximate CI developed in **H**.

Exercise 2

Exercise Rice 8: 8 (a) and (b) only.

Hint for (a): Use mle.

Hint for (b): Use section 8.5.3 – in particular the last paragraph before example **B**.
(Notice that there is a hidden application of Slutsky's lemma in Rice's argument).