HG
Oct. 2018

ECON 4130 H18

Exercises for seminar week 44

Exercise 1

This exercise is based on the ***Exam 2004H -"utsatt prøve"****, slightly extended and adapted to fit the present curriculum.*

The random variable (rv.), *Y*, has a log-normal distribution with parameters, μ and σ^2, if the density function (pdf) is given by

$$f(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma} \cdot \dfrac{1}{y} \cdot e^{-\frac{1}{2\sigma^2}[\ln(y)-\mu]^2} & \text{for} \quad y > 0 \\ 0 & \text{otherwise} \end{cases}$$

This is a right skewed distribution with a pdf somewhat similar to the pdf of a gamma distribution. It is sometimes used to model income distributions.

A. Show that, if *Y* is log-normal (μ, σ^2) then $X = \ln(Y)$ is normally distributed with expectation, μ and variance, σ^2 (i.e., $N(\mu, \sigma^2)$).

B. Explain how the moment generating function (mgf) for *X*, can be utilized to show that

$$E\left(Y^k\right) = e^{k\mu + k^2 \frac{\sigma^2}{2}} \qquad \text{for} \quad k = 1, 2, 3, \dots$$

C. The *variation coefficient* of a non-negative rv., *Z*, denoted by VC(*Z*), is defined as

$$\mathrm{VC}(Z) = \frac{\sqrt{\mathrm{Var}(Z)}}{\mathrm{E}(Z)}$$

The variation coefficient is a measure of variation. If Z is the income of a person randomly chosen from a population of income earners, VC(Z) is sometimes taken as a measure of income inequality for the population in question.

(i) Show that the VC is invariant for scale transformations, i.e., show that $\mathrm{VC}(cZ) = \mathrm{VC}(Z)$ for any constant $c > 0$.

(ii) Let Y be log-normal (μ, σ^2). Show that the variation coefficient, that we will denote by γ, is $\gamma = \mathrm{VC}(Y) = \sqrt{e^{\sigma^2} - 1}$

(iii) Let Y be gamma distributed, (α, λ), where α is the shape parameter and λ the scale parameter. Show that the variation coefficient is equal to $1/\sqrt{\alpha}$ and, hence, independent of the scale parameter λ.

D. Let $Z_1, Z_2, \ldots, Z_n$ be *iid* and non-negative rv's with expectation, $\mathrm{E}(Z_i) = \eta$, and variance, $\mathrm{Var}(Z_i) = \tau^2$. Otherwise we don't know anything about the common distribution of the Z_i's. Propose a consistent estimator for the VC in this case, and explain why it is consistent.

E. Let $Y_1, Y_2, \ldots, Y_n$ be *iid* and log-normally distributed (μ, σ^2). Show that the maximum likelihood estimators (MLE's) for μ and σ^2 are given by $\hat{\mu} = \overline{\ln(Y)} = \frac{1}{n}\sum_{i=1}^{n} \ln(Y_i)$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} [\ln(Y_i) - \overline{\ln(Y)}]^2$ respectively. [**Hint:** *Compare the log likelihood function with the log likelihood for a normal sample, i.e., study example B in Rice section 8.5*]

What is the MLE for the variation coefficient, $\gamma = \mathrm{VC}(Y_i)$?

F. Derive the moment estimators (MME's) for $\mu, \sigma^2,$ and γ, based on $Y_1, Y_2, \ldots, Y_n$ in question **E**.

G. We have a sample of n = 121 yearly incomes drawn from a population of women (Norway 1998) that is relatively homogenous with regard to the time spent at paid work.

Let Y_i denote the income of woman *i* in the sample. As before we assume that $Y_1, Y_2, \ldots, Y_n$ is *iid* and log-normally distributed (μ, σ^2). Calculate both the MLE- and MME estimates of the population VC, γ, based on the summary data in the table

Statistic	*Data*
n (sample size)	121
$\frac{1}{n}\sum_{i=1}^{n} Y_i$ (NOK)	202 799
$\frac{1}{n}\sum_{i=1}^{n} Y_i^2$	46 597 545 146
$\frac{1}{n}\sum_{i=1}^{n} \ln(Y_i)$	12.15916
$\frac{1}{n}\sum_{i=1}^{n} [\ln(Y_i)]^2$	147.96481

H. It can be shown that the MLE, $\hat{\sigma}^2$, is asymptotically normally distributed in the sense $\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \underset{n\to\infty}{\overset{D}{\rightarrow}} N(0, 2\sigma^4)$ (You do not need to show this here.) Use this to develop an asymptotic 95% confidence interval (CI) for γ based on a corresponding CI for σ^2. Calculate the interval.

I. A well known[1] fact is that if $X_1, X_2, \ldots, X_n$ are *iid* and normally distributed, $X_i \sim N(\mu, \sigma^2)$, then $\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2}$ is (exactly) χ^2 distributed with $n-1$ degrees of freedom. Use this to find an exact 95% CI for σ^2, and, from this, an exact 95% CI for γ. Calculate the interval and compare with the approximate CI developed in **H.**

Exercise 2

Exercise Rice 8: 8 (a) and (b) only.

[1] See, e.g., Rice section 6.3 (theorem B) for a proof (optional reading).

Hint for (a): Use mle.
Hint for (b): Use section 8.5.3 – in particular the last paragraph before example **B.** (Notice that there is a hidden application of Slutsky's lemma in Rice's argument).