HG
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ECON 4130

Exercises for seminar week 46

Exercise 1

For this exercise you need to have read *lecture notes* to Rice chapter 8 - especially section 0, 1, and 3.

a. Introduction on two-sided tests: Let θ be an unknown parameter in an econometric model, $\hat{\theta}$ an asymptotically normally distributed estimator based on *n* observations, and with a consistently estimated standard error[1]

$$\text{se}(\hat{\theta}) = \frac{\hat{b}}{\sqrt{n}}$$

such that

$$U_n = \frac{\hat{\theta}-\theta}{\text{se}(\hat{\theta})} = \sqrt{n}\,\frac{\hat{\theta}-\theta}{\hat{b}} \overset{\text{approximately}}{\sim} N(0,1) \quad \text{for any possible } \theta .$$

Based on U_n we have an approximate $1-\alpha$ confidence interval (CI) for θ given by

(1) $\hat{\theta} \pm z_{\alpha/2}\text{se}(\hat{\theta})$

where $z_{\alpha/2}$ is the upper $\alpha/2$ - quantile in *N*(0, 1) (i.e., such that $P(Z > z_{\alpha/2}) = \frac{\alpha}{2}$).

[1] In general, the standard error of an estimator, $\hat{\theta}$, of a parameter, θ, is defined as the square root of of the mean squared error (EMS), $\sqrt{E\left[(\hat{\theta}-\theta)^2\right]} = \sqrt{EMS}$. In particular, if $\hat{\theta}$ is unbiased, i.e., $E(\hat{\theta}) = \theta$, the standard error of $\hat{\theta}$ is the same as the standard deviation of $\hat{\theta}$ since, then, $\sqrt{EMS} \overset{Def}{=} \sqrt{E\left[\left(\hat{\theta}-\theta\right)^2\right]} = \sqrt{E\left[\left(\hat{\theta}-E(\hat{\theta})\right)^2\right]} \overset{Def}{=} \sqrt{\text{var}(\hat{\theta})} = SD(\hat{\theta})$. In practice, i.e., in computer outputs, the standard error, denoted by $SE(\hat{\theta})$, is a consistently estimated version of this. Also, in practice, when $\hat{\theta}$ is approximately (asymptotically) normally distributed with expectation, θ (i.e., $\hat{\theta}$ approximately unbiased), the standard error of $\hat{\theta}$ is usually understood to be the same as a consistently estimated standard deviation (i.e., the square root of the variance) in the approximatively normal distribution for $\hat{\theta}$.

Suppose we want to test the two-sided hypothesis, $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, where θ_0 is a known hypothetical value. Let $z_{\alpha/2}$ be the upper $\alpha/2$ - quantile in *N*(0, 1). An approximate α -level test is given by

(2) Reject H_0 if $W_n < -z_{\alpha/2}$ or $W_n > z_{\alpha/2}$ (i.e., if $|W_n| > z_{\alpha/2}$)

where $W_n = \dfrac{\hat{\theta} - \theta_0}{\mathrm{se}(\hat{\theta})}$ is the test statistic used. Note that the test having significance level approximately equal to α, means that

$$P(\text{Reject } H_0) = P(|W_n| > z_{\alpha/2}) \approx \alpha \qquad \text{if } \theta_0 \text{ is the true value of } \theta.$$

Question: Show that the test criterion (2) is equivalent to the following test criterion based on the CI in (1):

(3) Reject H_0 if θ_0 lies outside the CI,
(which, thus, represents an alternative way to perform the two-sided test).

[**Hint**: If *L* and *U* denote the lower and upper limit in the CI (1) respectively, show that the criterion (2) is equivalent with: $\theta_0 < L$ or $\theta_0 > U$. Notice also that if the true value of θ is θ_0, then

$$P(\text{The interval } (L,U) \text{ does not cover } \theta_0) = 1 - P(L \leq \theta_0 \leq U) \approx 1 - (1-\alpha) = \alpha.]$$

[**Note NB!** Hence a CI of the form (1) can always be used to test two-sided hypotheses about θ. Not only that – we get something in addition: If we reject H_0 by the test in (2), we may conclude (with strong evidence) not only that $\theta \neq \theta_0$, but also on which side of θ_0 the unknown θ lies. For example, if θ_0 lies outside to the left of the CI, we may conclude not only that $\theta \neq \theta_0$, but also (with equally strong evidence) that $\theta > \theta_0$. This is simply because the CI itself shows that the true θ then (with strong evidence) lies to the right of θ_0.

(This procedure may also be justified "deeper" using *statistical decision theory* that handles test situations where there are several different alternative hypotheses to a single null-hypothesis – not treated in this curriculum. I.e., contrary to the classical formulation of the problem which only allows one alternative, $\theta \neq \theta_0$, to the null-hypothesis, we now prefer to operate with two alternatives, $H_{11} : \theta < \theta_0$ and $H_{12} : \theta > \theta_0$ to the null-hypothesis – which often appears to be a more proper formulation of the problem and leads to the interpretation given).]

b. Let $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$ be a vector of three unknown parameters in an econometric model.

Suppose that some estimating principle has produced an approximately normally distributed estimator $\hat{\beta}' = (\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)$ such that

$$\hat{\beta} \overset{\text{approximately}}{\sim} N(\beta, \Sigma)$$

where Σ is (a consistent estimate of) the covariance matrix given by

$$\Sigma = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 4 & 1 \\ 3 & 1 & 16 \end{pmatrix}$$

(so, we may proceed, using the approximate distribution, *as if* the covariance matrix of $\hat{\beta}$ were known – and the standard errors will simply be the square root of the variances.)

(i) Suppose further that the estimates of the β_j 's are given in the table. Fill in the standard errors in the table, and calculate the three correlation coefficients, $\text{corre}(\hat{\beta}_1, \hat{\beta}_2)$, $\text{corre}(\hat{\beta}_1, \hat{\beta}_3)$, and $\text{corre}(\hat{\beta}_2, \hat{\beta}_3)$ (based on the approximate model).

Table 1

Coefficient	β_1	β_2	β_3
Estimate	10	15	16
Standard error	?	?	?

(ii) Calculate an approximate 95% CI for β_1, and perform a test (with level of significance 5%) of $H_0 : \beta_1 = 5$ versus $H_1 : \beta_1 \neq 5$.

c. Introduce the parameters

$$\theta_1 = \beta_1 - \beta_2$$
$$\theta_2 = \beta_1 - \beta_3$$
$$\theta_3 = \frac{\beta_2 + \beta_3}{2}$$
$$\theta_4 = \beta_1 - \theta_3$$

Produce a table as **Table 1** with estimates (based on $\hat{\beta}$) and standard errors for the three parameters $\theta_1, \theta_2,$ and, θ_4.

d. Test the following three two-sided hypotheses (level 5%) by using CI's

(i) $H_0 : \beta_1 = \beta_2$ vs $H_1 : \beta_1 \neq \beta_2$ (i.e., $H_0 : \theta_1 = 0$ vs $H_1 : \theta_1 \neq 0$)
(ii) $H_0 : \beta_1 = \beta_3$ vs $H_1 : \beta_1 \neq \beta_3$
(iii) $H_0 : \beta_1 = \theta_3$ vs $H_1 : \beta_1 \neq \theta_3$

In each case where the test leads to rejection of H_0, state the direction of the alternative in the conclusion (for example, if the test of (i) leads to rejection, state one of "$\beta_1 < \beta_2$" or "$\beta_1 > \beta_2$" as your conclusion instead of just "$\beta_1 \neq \beta_2$").

Exercise 2

Let *X* be the number of traffic accidents occurring during *t* months in a region. Assume that *X* is poisson distributed with parameter λt (i.e., $X \sim \text{pois}(\lambda t)$).

a. Explain why the parameter λ can be interpreted as a theoretical incidence rate, i.e., the expected number of accidents per month.

b. We cannot observe λ directly, but we can observe *X* instead. Show that the estimator $\hat{\lambda} = X/t$

i) is unbiased for all t,
ii) is consistent as $t \to \infty$ (use Chebyshev's inequality).

c. Using the fact that *X* is approximately normally distributed when λt is large (≥ 10 is usually considered sufficient), develop an approximate $1-\alpha$ confidence interval (CI) for λ based on *X*.

[**Hint:** Show first, using Slutsky's lemma, that

$$\sqrt{t}\,\frac{\hat{\lambda}-\lambda}{\sqrt{\hat{\lambda}}} \underset{t\to\infty}{\overset{D}{\to}} Z \sim N(0,1)$$

Note also that $\sqrt{t}\,\dfrac{\hat{\lambda}-\lambda}{\sqrt{\lambda}} = \dfrac{X-\lambda t}{\sqrt{\lambda t}}$]

Exercise 3

We are interested in the monthly incidence rate of traffic accidents in Norway. From Statistical Office Norway (SSB), we obtain the number of traffic accidents registered in the period 2003 – 2005, as given in table 1,

Table 1

Year	No. of traffic accidents
2003	8266
2004	8425
2005	8078
Sum	24769

We want a 95% CI for the monthly incidence rate based on these results. Let X_1, X_2, X_3 denote the rv's behind the three observations in table 1. Our first approach is to calculate a "t-interval" for the incidence rate, called λ , based on the following model

Model 1 X_1, X_2, X_3 are *iid* with $X_i \sim N(\mu, \sigma^2)$ where $\mu = 12\lambda$

[**Hint:** When $X_1, X_2, \ldots, X_n$ are *iid* with $X_i \sim N(\mu, \sigma^2)$, we remember from the basic statistic course that an (exact) $1-\alpha$ CI for μ (the so called "t-interval") is

$$\bar{X} \pm t_{1-\alpha/2,\ n-1} \cdot \frac{S}{\sqrt{n}}, \qquad \text{where } \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \text{ and}$$

where $t_{1-\alpha/2,\ n-1}$ is the $1-\alpha/2$ percentile in the t-distribution with $n-1$ degrees of freedom.]

a. Calculate the 95% CI for μ based on model 1 and transform the interval to a corresponding CI for λ. Explain why the interval for λ must have the same degree of confidence as the one for μ. Discuss briefly whether the assumptions in model 1 appear reasonable or not.

b. An alternative approach is to assume that the total number of registered accidents in the period 2003 - 2005, $X = X_1 + X_2 + X_3$, is poisson distributed, i.e.,

Model 2 $X \sim \text{pois}(t\lambda)$ with $t = 36$

Calculate an approximate 95% CI for λ based on model 2, and compare with the CI in **a**.

c. One reasonable criticism that can be raised against the model 2, is the apparently unrealistic assumption of *constant incidence rate* that underlies the poisson model, which, among other things, implies that all the months of the year have the same incidence rate, λ. For example, table 2, that gives the number of accidents for January and June, appears to support this criticism.

Table 2

	Number of accidents	
Year	January	June
2003	576	805
2004	616	847
2005	588	853

Luckily, the poisson model offers an easy way to accommodate this criticism. To see this, first prove the following result [**Hint:** Use the mgf for the poisson distribution]:

Property 1 Let $Y_1, Y_2, \ldots, Y_k$ be independent and poisson distributed with $Y_j \sim \text{pois}(\mu_j)$ for $j = 1, 2, \ldots, k$. Then $Y = Y_1 + Y_2 + \cdots + Y_k \sim \text{pois}(\mu)$ where $\mu = \mu_1 + \mu_2 + \cdots + \mu_k$.

d. In order to accommodate the criticism, we suggest the following model: Let Y_{ij} be the number of accidents in month *j* ($j = 1, 2, \ldots, 12$) in year *i* (*i* = 1,2 3). Assume

Model 3 The Y_{ij}'s are independent and poisson distributed with $Y_{ij} \sim \text{pois}(\lambda_j)$ for $j = 1, 2, \ldots, 12$ and $i = 1, 2, 3$.

Show that $X = \sum_{i=1}^{3} \sum_{j=1}^{12} Y_{ij} \sim \text{pois}(36\bar{\lambda})$ where $\bar{\lambda} = \frac{1}{12} \sum_{j=1}^{12} \lambda_j$ is the average monthly incidence rate.

e. Show that the estimator in exercise 2, $\hat{\lambda} = \frac{X}{t}$, where $t = 12r$ is the number of months and *r* the corresponding number of years, is unbiased for $\bar{\lambda}$ and with variance, $\operatorname{var}(\hat{\lambda}) = \frac{\bar{\lambda}}{t}$. This shows that $\hat{\lambda}$ also is consistent for $\bar{\lambda}$ as $t \to \infty$ (why?).

f. Explain why the CI in **b** is still valid, but now as an approximate 95% CI for the new parameter, $\bar{\lambda}$.