Econ 4130
HG Oct. 2018

Lecture note on the interpretation of regression coefficients

1) The effect of *X* in the simple linear regression model

To fix ideas, let Y = consumption of a certain class of goods and X = income, for a randomly chosen individual from the population. (X, Y) is jointly distributed with a joint pdf, $f(x, y)$ (the population distribution). The dependence of the response, Y, on the explanatory variable, X, is usually studied by means of the conditional distribution of Y for fixed values of X (i.e., $X = x$), with pdf, $f(y \mid x) = f(x, y)/f_1(x)$, where the marginal pdf for X, is $f_1(x) = \int_{-\infty}^{\infty} f(x, y)dy$. The regression function is simply the expected value of $(Y \mid X = x)$ in $f(y \mid x)$, i.e., $\mu(x) = E(Y \mid x) = \int_{-\infty}^{\infty} yf(y \mid x)dy$, which expresses the expected response for a given fixed value, $X = x$. In the simple linear regression model we postulate that $\mu(x)$ is a linear function

(1) $$\mu(x) = E(Y \mid x) = \alpha + \beta x$$

In this model the regression coefficient, β, can be interpreted as the effect of a unit change of X (i.e., $(X = x) \to (X = x+1)$) on the expected change of the response, Y.

> **Elaboration.** Let Y_1 be the consumption for a randomly chosen individual with income, $X = x$, and Y_2 correspondingly for a randomly chosen individual with $X = x+1$. Then the pdf's of Y_1, and Y_2 are $f(y \mid x)$ and $f(y \mid x+1)$ with expected values, $\mu(x)$ and $\mu(x+1)$, respectively. The expected difference becomes β since
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> $$E(Y_2 - Y_1) = E(Y_2) - E(Y_1) = \mu(x+1) - \mu(x) = \alpha + \beta(x+1) - \alpha - \beta x = \beta$$

Note how the interpretation of β is derived from the meaning of the function, $\mu(x) = E(Y \mid x)$.

Note also that this interpretation *does not* apply to a single individual. It does not say anything about the expected response when a *single* individual increases the income from $X = x$ to $X = x+1$. For getting information on such effects we will need at least two observations of X and Y for each individual at two different points in time (i.e., panel data).

If we want the effect of 10 (say) units change in X, the same calculation gives

$$\mu(x+10) - \mu(x) = 10\beta .$$

2) The effect of *X*, controlling for *Z* (wealth)

Now consider *Z* (e.g., wealth) as an additional explanatory variable that may influence *Y*. We want to find the effect on the expected response of a unit change of *X* - *controlling for Z.* Suppose the postulated regression function is

(2) $\mu(x,z) = E(Y \mid x,z) = \alpha + \beta x + \gamma z$

which is the expectation in the conditional distribution of *Y* for fixed values of $X = x$ and $Z = z$, with pdf,

$f(y \mid x,z) = \dfrac{f(x,y,z)}{f_1(x,z)}$, where the marginal pdf of (X,Z) is $f_1(x,z) = \int_{-\infty}^{\infty} f(x,y,z)dy$,

and where $f(x,y,z)$ is the joint pdf of (X,Y,Z).

We are now interested in the expected difference between two rv's, Y_1, Y_2 (as in the elaboration under 1), where

- Y_1 is the consumption for a randomly chosen individual with income, $X = x$ and Z=z (i.e., $Y_1 = (Y \mid X = x, Z = z)$)
- Y_2 is the consumption for a randomly chosen individual with income, $X = x+1$ and Z=z (i.e., $Y_2 = (Y \mid X = x+1, Z = z)$).

Notice that Y_1 and Y_2 both have the same value, *z*, of *Z* (which is what we mean by "controlling for *Z*"). This is, of course, to make the comparison between Y_1 and Y_2 more fair. Then, the expected difference becomes

(3) $E(Y_2 - Y_1) = E(Y_2) - E(Y_1) = \mu(x+1,z) - \mu(x,z) = \alpha + \beta(x+1) + \gamma z - \alpha - \beta x - \gamma z = \beta$

Thus, β can be interpreted as the expected change in the response (*Y*) between two subpopulations of individuals where all individuals in the first subpopulation have $X = x$ and all individuals in the other have $X = x+1$, and where all individuals in both groups *have the same value* of the wealth ($Z = z$). This is often expressed by saying that β is "the effect of a unit change of X on (expected) *Y*, *ceteris paribus* – which translates to "everything else equal." Alternatively, β is sometimes called "*the partial effect* of a unit change in *X* (controlling for other explanatory variables)".

An advantage with this particular model is that the cet. par. effect of *X* reduces to a single parameter (β) no matter what the wealth (*Z*) is.

[Notice, in passing, that if the values of *Z* were different for Y_1 and Y_2, e.g., $Z = z_1$ for Y_1 and $Z = z_2$ for Y_2, the calculation in (3) gives,

$E(Y_2 - Y_1) = E(Y_2) - E(Y_1) = \mu(x+1, z_2) - \mu(x, z_1) = \beta + \gamma(z_2 - z_1)$, which shows that the effect of a unit change in X - in that case - is partly due to differences in the wealth (if $\gamma \neq 0$, of course).]

3) The effect of *X*, controlling for *Z* (wealth) and *V* (age)

Now we look at the conditional distribution of *Y* for fixed values, $X = x, Z = z, V = v$, with pdf $f(y \mid x, z, v) = \dfrac{f(x, y, z, v)}{f_1(x, z, v)}$, where the marginal pdf of (X, Z, V) is $f_1(x, z, v) = \int_{-\infty}^{\infty} f(x, y, z, v)\,dy$. The expectation in this distribution is a function of *x*, *z*, and *v*, $\mu(x, z, v) = E(Y \mid x, z, v)$

The effect of a unit change in *X* (ceteris paribus), can be calculated as above

$$\mu(x+1, z, v) - \mu(x, z, v)$$

In the special case that we postulate a linear regression model, $\mu(x, z, v) = \alpha + \beta x + \gamma z + \delta v$, this calculation gives us

$$\mu(x+1, z, v) - \mu(x, z, v) = \alpha + \beta(x+1) + \gamma z + \delta v - (\alpha + \beta x + \gamma z + \delta v) = \beta$$

so the cet. par. effect of a unit change in X (sometimes also called "the income-effect on consumption") reduces to a single parameter, β, no matter what the wealth and age are.

4) Modelling interaction between *X* (income) and *Z* (wealth)

It is imaginable that the income-effect on consumption is different between rich and poor people. If this is the case, we say there is an *interaction* between income and wealth. An easy way to model this is to include the product term, *xz* (also called an interaction term), in the regression function

$$E(Y \mid x, z) = \mu(x, z) \overset{\text{postulate}}{=} \alpha + \beta x + \gamma z + \partial xz$$

Note that although this regression function is a non-linear function of *x* and *z*, we still call it *a linear regression model* since it is linear in the parameters, $\alpha, \beta, \gamma,$ and δ. Being linear in the parameters implies that it can be estimated by usual least squares techniques (e.g., OLS in the case of homoscedasticity, i.e., when we can postulate that $\text{var}(Y \mid x, z) =$ constant))[1].

The cet. par. effect of a unit change in *X* can be calculated as before

$$\mu(x+1, z) - \mu(x, z) = \alpha + \beta(x+1) + \gamma z + \delta(x+1)z - (\alpha + \beta x + \gamma z + \delta xz) = \beta + \partial z$$

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[1] If you wish to estimate this model by Stata (say), you need 4 variables, each with n observations, in the Stata data matrix: the response y, and 3 explanatory variables, x1 = x, x2 =z, and x3 =xz. Then the following stata command for OLS does it: `regr y x1 x2 x3`

Hence (e.g.), if $\delta < 0$, the income-effect on consumption will be smaller for rich than for poor people.

If there are several explanatory variables, the price for including all sorts of interactions in the model is a large number of extra parameters in the regression function. For example, if we include *V*=age as an explanatory variable, a full interaction regression function could look like

$$\mu(x,z,v) = \alpha + \beta_1 x + \beta_2 z + \beta_3 v + \gamma_1 xz + \gamma_2 xv + \gamma_3 zv + \gamma_4 xzv$$

The interaction term, *xzv*, is called a 2nd order interaction term. Check yourself that the cet. par. effect of a unit change of *X*, now becomes,

$$\mu(x+1,z,v) - \mu(x,z,v) = \beta_1 + \gamma_1 z + \gamma_2 v + \gamma_4 zv$$

5) The income effect on consumption may also depend on income

Consider now, for simplicity, only *X* (income) as explanatory. Postulate a regression function

$$E(Y \mid x) = \mu(x) = \alpha + \beta x + \gamma x^2$$

(This is also a linear regression model that may be well estimated by OLS under the assumption of homoscedasticity since it is linear in the parameters, α, β, γ .)[2]

The income-effect on consumption now becomes

$$\mu(x+1) - \mu(x) = \alpha + \beta(x+1) + \gamma(x+1)^2 - (\alpha + \beta x + \gamma x^2) = \beta + \gamma(2x+1)$$

Hence (e.g.), if $\gamma < 0$, the income effect is smaller for high-income people than for low-income people in this model.

[2] To estimate this model in Stata (say), you need 3 variables (each with n observations) in the data matrix: the response y, and 2 explanatory variables, x1 = x, x2 = x^2, and the OLS command becomes, `regr y x1 x2`