E4130 2018 Tu 12/11-18 (23.1)

MM Rice 9.4 – 9.6

Likelihood ratio testing (LR)

$\theta' = (\theta_1, \ldots, \theta_r)$ unknown.

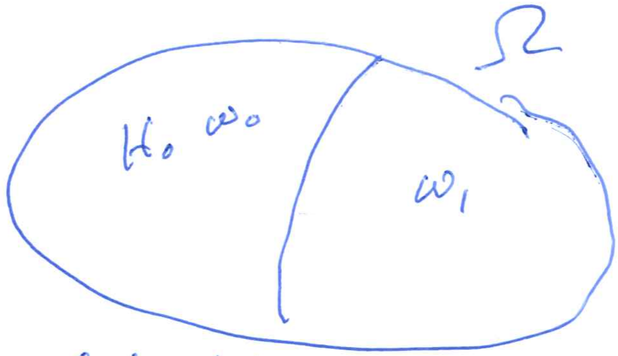
A priori $\theta \in \Omega$ (full model)

large omega

Reduced model

$H_0 : \theta \in \omega_0$ ← small omega $\quad (\omega_0 \subseteq \Omega)$

$H_1 : \theta \in \omega_1 \quad (\omega_1 = \Omega - \omega_0)$



Let $L(\theta)$ the likelihood

H_0: maximize L in ω_0

$$L_{\omega_0} = \max_{\theta \in \omega_0} L(\theta) = L(\tilde{\theta})$$

$\tilde{\theta}$ is mle under ω_0

Ω: — " — in Ω

max value $L_\Omega = \max_{\theta \in \Omega} L(\theta) = L(\hat{\theta})$

$\hat{\theta}$ is mle under Ω

Likelihood ratio:

$$\Lambda = \frac{L_{\omega_0}}{L_\Omega} = \frac{L(\tilde{\theta})}{L(\hat{\theta})} = \frac{\max_{\omega_0} L(\theta)}{\max_{\Omega} L(\theta)}$$

capital lambda

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Note $\Lambda \le 1$ always ($\omega_0 \subseteq \Omega$)

If Λ is small enough

(i.e. far from 1) $\Rightarrow$ evidence against ω_0

H_0

Note: $-2 \ln \Lambda$ is approx. χ^2-distr. under H_0

Small Λ $\Leftrightarrow$ $\ln \Lambda$ small (near $-\infty$)

$\Leftrightarrow$ $-2 \ln \Lambda$ large positive.

Basic thm

$$-2 \ln \Lambda \overset{\text{approx.}}{\sim} \chi^2_{df} \quad \text{under } H_0 \ (\theta \in \omega_0)$$

for "large" n

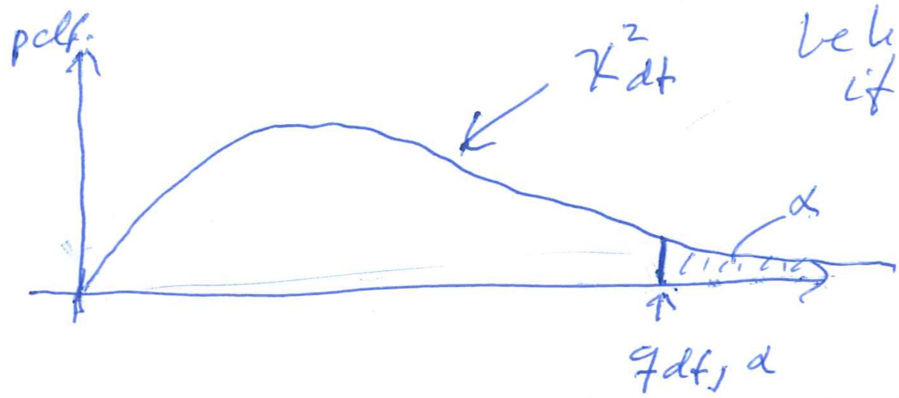
where $df = \dim \Omega - \dim \omega_0$

no. of free parameters in Ω — no. of free parameters in ω_0

no of restrictions on $\theta' = (\theta_1, \dots, \theta_r)$

So reject H_0 if

$W = -2\ln\Lambda$ is large enough.



behaviour of W if H_0 is true.

(upper α-quantile in χ^2_{df})

χ^2-test (level of significance α)

Reject H_0 if $-2\ln\Lambda \geq q_{df,\alpha}$
(critical value)

where $P_{H_0}(W \geq q_{df,\alpha}) = \alpha$
$\simeq \alpha$

or p-value: $P_{H_0}(W \geq w_{obs})$

w_0 = observed W

Two ways of determining df

$df = \dim \Omega - \dim \omega_0$
$=$ no. of restrictions to get ω_0 from Ω.

A linear restriction on $\theta_1, \ldots, \theta_r$ has general form:

$$a_0 + a_1\theta_1 + a_2\theta_2 + \cdots + a_r\theta_r = 0$$

where $a_0, a_1, \ldots, a_r$ are known constants.

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Ex. 1. $X_1, \ldots, X_n \sim f(x_1, \ldots, x_n, \theta)$

$\theta' = (\theta_1, \ldots, \theta_s, \ldots, \theta_r)$

$H_0: \theta_1 = \theta_2 = \cdots = \theta_s = 0 \quad (s < r)$

s restrictions: $(\theta_1 = 0, \theta_2 = 0, \ldots, \theta_s = 0)$

$\dim \Omega = r$

Let η' be the parameter vector under H_0

$\eta' = (\theta_{s+1}, \theta_{s+2}, \ldots, \theta_r) \Rightarrow \dim \omega_0 = r - s$

$df = \dim \Omega - \dim \omega_0$
$= r - (r - s) = s$

Special: H_0: $f(x_1, \ldots, x_n)$ fully known
$\Rightarrow \dim \omega_0 = 0$

$\dim \Omega = r \Rightarrow df = r - 0 = r$

Ex 2: $\theta' = (\theta_1, \dots, \theta_r)$ $\quad \dim \Omega = r$

H_0: $\theta_1 = \theta_2 = \theta_3$ (put μ for the common value)

$$\left.\begin{array}{l} \theta_1 - \theta_2 = 0 \\ \theta_2 - \theta_3 = 0 \end{array}\right\} \iff H_0 \quad \text{(2 restrictions)}$$

so $s = 2 = df$

η' = param. vector under H_0

$$\eta' = (\underset{1}{\mu}, \underbrace{\theta_4, \theta_5, \dots, \theta_r}_{r-3})$$

$\dim \omega_0 = r - 3 + 1 = r - 2$

$df = r - (r-2) = 2$

Ex 3. H_0: $\theta_2 = \frac{1}{2}\theta_1$

$\iff \theta_2 - \frac{1}{2}\theta_1 = 0$

(i.e. $s=1$ restriction)

H_0: $\theta_3 = \dfrac{\theta_1 + \theta_2}{2}$

$\iff \theta_3 - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2 = 0$

i.e. $df = 1$ restriction.

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Ex 4. H_0: $\theta_2 = 6$ and $\theta_3 = 7$

2 restrictions

$\theta_2 - 6 = 0 \qquad \theta_3 - 7 = 0$

Ex 5: $\underbrace{\theta_1 = \theta_2 = \theta_3}_{\text{2 restr.}}$ and $\underbrace{\theta_5 = 10}_{\text{1 restr.}}$

$df = 2 + 1 = 3$

Multinomial case.

Ω: $(X_1, \dots, X_m) \sim \text{multin}(n, p_1, \dots, p_m)$

$\sum p_j = 1$

$\sum_{i=1}^{m} X_i = n$

We found MLE's under Ω

$\hat{p}_j = \frac{X_j}{n} \qquad j = 1, \dots, m$

H_0: $p_j = p_j(\theta_1, \dots, \theta_r) = p_j(\theta)$

mle: $\tilde{p}_j = p_j(\hat{\theta}) \qquad \hat{\theta}$ mle under H_0.

LR: $\dfrac{\cancel{\frac{n!}{x_1! \cdots x_m!}} \tilde{p}_1^{x_1} \tilde{p}_2^{x_2} \cdots \tilde{p}_m^{x_m}}{\cancel{\frac{n!}{x_1! \cdots x_m!}} \hat{p}_1^{x_1} \hat{p}_2^{x_2} \cdots \hat{p}_m^{x_m}} = \dfrac{L(\tilde{p})}{L(\hat{p})} = \Lambda$

$W = -2 \ln \Lambda = -2 \ln \left[\left(\frac{\tilde{p}_1}{\hat{p}_1}\right)^{x_1} \left(\frac{\tilde{p}_2}{\hat{p}_2}\right)^{x_2} \cdots \left(\frac{\tilde{p}_m}{\hat{p}_2}\right)^{x_m} \right]$

$= -2 \sum_{j=1}^{m} \underbrace{x_j}_{O_j} \ln \frac{\tilde{p}_j}{\hat{p}_j} = -2 \sum_{j=1}^{m} x_j \ln \frac{n \tilde{p}_j}{n \hat{p}_j}$

$O_j = x_j = n \cdot \hat{p}_j$

$E_j = n\tilde{p}_j$ (expected freq. under H_0)

$$W = -2 \sum_{j=1}^{m} O_j \ln \frac{E_j}{O_j} \underset{H_0}{\overset{\text{approx}}{\sim}} \chi^2_{df}.$$

$$df = \dim \Omega - \dim \omega_0 = m - 1 - r$$

$$\text{Pearson } Q = \sum_{j=1}^{m} \frac{(O_j - E_j)^2}{E_j} \underset{H_0}{\overset{\text{approx.}}{\sim}} \chi^2_{m-1-r}$$

It turns out (by Taylor expansion 2 terms. on Λ) that W is asymptotically equivalent with Pearson.

(i.e., $W - Q \xrightarrow[n\to\infty]{P} 0$

$$\Rightarrow W = \underset{\xrightarrow{D} \chi^2_{df}}{Q} + \underbrace{(W - Q)}_{\xrightarrow{P} 0} \xrightarrow{D} \chi^2_{df}.)$$

Ex: $n = 50$ call-length. $H_0: Y_i \sim \exp(\lambda)$

cat.	$O_j = X_j$	$E_j = n\tilde{p}_j(\hat{\lambda})$	$\frac{(O_j - E_j)^2}{E_j}$	$-2\, O_j \ln \frac{E_j}{O_j}$
0-5	17	24.2	2.15	-12.02
5-10	21	12.5	5.80	21.82
10-15	3	6.4	0.70	3.47
>15	4	7.0	1.19	-4.32
			$Q = 9.52$	$W = 8.96$

$$\begin{matrix} Q \\ W \end{matrix} \underset{H_0}{\overset{\text{appr.}}{\sim}} \chi^2_{3-1} = \chi^2_2$$

Traffic accidents. (week 46 seminar)

23.8

Year	jan. X_j
2003	576
04	616
05	588
sum	1780

Full model. $X_j \sim pois(\lambda_j)$ $j=1,2,3$ (λ_j: EX_j)

X_1, X_2, X_3 independent.

$H_0: \lambda_1 = \lambda_2 = \lambda_3$

df = no. of restrictions $= 2$

$$\Omega: f(x_1, x_2, x_3) = \prod_{j=1}^{3} P(X_j = x_j)$$

$$= P(X_1 = x_1) P(X_2 = x_2) P(X_3 = x_3)$$

$$= \prod_{j=1}^{3} \frac{\lambda_j^{x_j}}{x_j!} e^{-\lambda_j} = L(\lambda_1, \lambda_2, \lambda_3)$$

$$l = \ln f = \sum_{j=1}^{3} \ln\left(\frac{\lambda_j^{x_3}}{x_3!} e^{-\lambda_j}\right)$$

$$= \sum_{j=1}^{3} \left[x_j \ln \lambda_j - \lambda_j - \ln x_3! \right]$$

$$\frac{\partial l}{\partial \lambda_j} = \frac{x_j}{\lambda_j} - 1 = 0 \qquad j=1,2,3 \text{ and } \lambda = \hat{\lambda}_j$$

$$\Rightarrow \hat{\lambda}_j = x_j \qquad (O_j)$$

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Under H_0 $(\lambda_1 = \lambda_2 = \lambda_3 \ (= \lambda_0))$

Show yourself: $\hat{\lambda} = \dfrac{X_1 + X_2 + X_3}{3} = \bar{X}$

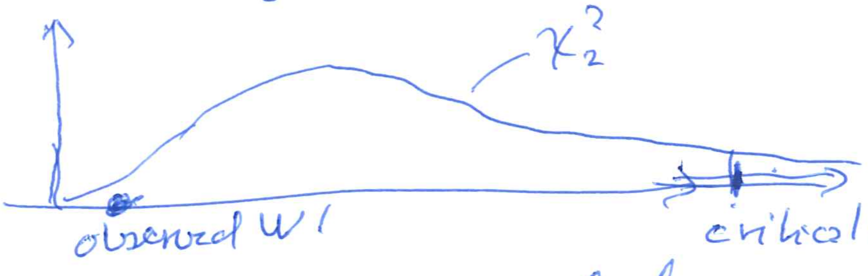
$$\Lambda = \frac{\prod_{j=1}^{3} \frac{1}{x_j!} \hat{\lambda}^{x_j} \cdot e^{-\hat{\lambda}}}{\prod_{j=1}^{3} \frac{1}{x_j!} \hat{\lambda}_j^{x_j} e^{-\hat{\lambda}_j}}$$

$$W = -2 \ln \Lambda = -2\left[\sum_{j=1}^{3} x_j \ln \frac{\hat{\lambda}}{\hat{\lambda}_j} - \underbrace{\sum_{j=1}^{3} (\hat{\lambda} - \hat{\lambda}_j)}_{3\hat{\lambda} = \Sigma x_j \quad \Sigma x_j \quad = 0}\right]$$

$$= -2 \sum_{j=1}^{3} x_j \ln \frac{\hat{\lambda}}{\hat{\lambda}_j} \ \underset{H_0}{\overset{\text{approx}}{\sim}} \ \chi^2_2$$

$O_j = X_j$	$-2 X_j \cdot \ln \frac{\hat{\lambda}}{\hat{\lambda}_j}$
576	0.059
612	-0.075
588	0.018
sum	0.002

P value: $P_{\chi^2_2}(-2\ln\Lambda > 0.002) \approx 0.999$



No evidence in data to doubt H_0