HG
Sep. 2019

Exercises for seminar week 40

Introductory note for exercise 3:

You will need the following important theorem (that we proved in the lecture for the sum of n = two independent random variables) about moment generating functions (mgf's) for sums of independent variables:

Theorem (a slightly more general version of theorem 4 given in the lecture on mgf's.)

If $X_1, X_2, \ldots, X_n$ are independent rv's (discrete or continuous), with individual mgf's, $M_j(t) = E\left[e^{tX_j}\right], \quad j = 1, 2, \ldots, n$, and $Y = X_1 + X_2 + \cdots + X_n$, then the mgf of the sum Y is $M_Y(t) = M_1(t)M_2(t)\cdots M_n(\text{t})$.

The proof of this more general result is quite similar to the proof for $n = 2$, presented in the lecture, but uses a general proof technique called *induction proof* that we do not assume known in this course. However, you should know the result of the theorem, and be able to use it in exercises.

Example 1. A sum of independent normally distributed variables is (exactly) normally distributed: Let $X_1, X_2, \ldots, X_n$ be independent and normally distributed (with possibly different normal distributions), where $X_j \sim N(\mu_j, \sigma_j^2), \quad j = 1, 2, \ldots, n$. Then the theorem shows us that

$$Y = X_1 + X_2 + \cdots + X_n \quad \sim \quad N\left(E(Y), \text{var}(Y)\right) = N\left(\mu_1 + \mu_2 + \cdots + \mu_n,\ \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2\right)$$

because:

In the lectures we derived the mgf of X_j as $M_j(t) = e^{\mu_j t + \frac{\sigma_j^2}{2}t^2}$, $j = 1, 2, \ldots, n$. The theorem then gives us

$$M_Y(t) = M_1(t)M_2(t)\cdots M_n(\text{t}) = e^{\mu_1 t + \frac{\sigma_1^2}{2}t^2} \cdot e^{\mu_2 t + \frac{\sigma_2^2}{2}t^2} \cdot \cdots \cdot e^{\mu_n t + \frac{\sigma_n^2}{2}t^2} =$$

$$= e^{t(\mu_1 + \mu_2 + \cdots + \mu_n) + \frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}{2} \cdot t^2}$$

which is the mgf of the $N\left(\mu_1 + \mu_2 + \cdots + \mu_n,\ \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2\right)$ distribution. The uniqueness of the mgf (cf. theorem 3 from the lecture) then implies that Y must have this normal distribution.

Example 2. A sum of independent gamma distributed variables with the same scale, λ, but possibly different α_j's is (exactly) gamma distributed: Let $X_1, X_2, \ldots, X_n$ be independent and gamma distributed, where $X_j \sim \Gamma(\alpha_j, \lambda), \quad j = 1, 2, \ldots, n$. Then the theorem shows us that $Y = X_1 + X_2 + \cdots + X_n \quad \sim \quad \Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n,\ \lambda)$ **because**:

In the lectures we derived the mgf of X_j as $M_j(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_j}$, $j = 1, 2, \ldots, n$. The theorem then gives us

$$M_Y(t) = M_1(t)M_2(t)\cdots M_n(\text{t}) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1}\left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2}\cdots\left(\frac{\lambda}{\lambda - t}\right)^{\alpha_n} =$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}$$

which is the mgf of the $\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n,\ \lambda)$ distribution. The uniqueness of the mgf then gives us $Y = X_1 + X_2 + \cdots + X_n \quad \sim \quad \Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n,\ \lambda)$.

Note that, if $n = 2$, and the two scales, λ_1, λ_2, are different, $Y = X_1 + X_2$ is *not* gamma distributed since the mgf becomes , $M_Y(t) = \left(\frac{\lambda_1}{\lambda_1 - t}\right)^{\alpha_1}\left(\frac{\lambda_2}{\lambda_2 - t}\right)^{\alpha_2}$, which is the mgf of some other distribution than gamma.

On the other hand, you may prove, as an exercise, using mgfs, that $V = \lambda_1 X_1 + \lambda_2 X_2$ is in fact gamma distributed…

End of introduction.

Exercise 1 Rice chap. 2, no. 59

[**Hint.** Use a similar argument as in example 2.3C (page 61) in Rice.]

Exercise 2 Rice chap. 4, no. 66

[**Hint.** Remember that the law of total expectation is still valid (can be proven) if one rv of (X, Y) is discrete and the other is continuous.]

Exercise 3 Rice chap. 4, no. 81 and 82

[**Hint for 82.** Remember that, if X is binomial (n, p) distributed, then *X* can be written as a sum, $X = \sum_{i=1}^{n} X_i$, where $X_1, X_2, \ldots$ are *iid* Bernoulli random variables.]

Exercise 4 Problem 2 in postponed exam 2006H (reproduced here):

[**Note.** A printing mistake in question **a** has been corrected.]

[**Hint** for question **d**: Utilize the mgf of *T*]

[**Note.** An extra hint has been added to question **e**.]

Poor people in the third world have seldom the opportunity to obtain loans in the bank, or even a bank account, and have great difficulties to save for durable consumer goods. In the slum area in Nairobi by the name of Kibera many have organized themselves in a manner that actually makes it easier to save. One type of organization is called ROSCA (Rotating Saving and Credit Association), which is usually organized as follows:

> A group of n poor people, mostly women, agree to establish a mutually binding lottery that lasts for one year. Every member commits herself to participate in the lottery for the whole year. The year is divided into n periods. In each period every member contributes an amount, s/n, to a common pool (also called pot) which, at the end of each period, thus, contains the total amount, s. At the end of the period the group comes together and chooses by lottery one person among those that have not already been chosen in earlier periods. In the lottery everyone eligible have the same chance to be chosen. The chosen person then gets the total amount, s, of the pot for disposal. We assume here, at least initially, that no member fails her commitment and drops out of the lottery immediately after having been paid the amount s. Hence every member contributes a total of s distributed over n payments during the year.

In this way each participant gets hold of the sum s at a random point in time during the year without having to save by hiding money in the mattress or other places where it is likely that the husband or others may find the money and disappear with it.

a. Imagine that you are one of the participants of a ROSCA, and suppose that you win the pot after V periods. The lottery implies that V is a discrete random variable with a uniform distribution over the integers, 1,2,…,n, i.e.,

$$P(V=v)=\frac{1}{n} \quad \text{for } v=1,2,\ldots,n$$

Show that $E(V)=\dfrac{n+1}{2}$ and $\operatorname{var}(V)=\dfrac{n^2-1}{12}$.

[**Hint:** The following formulas may be useful:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$]

b. Let Y be the waiting time, *measured in years*, until you win the pot. Explain why

$$E(Y)=\frac{n+1}{2n}$$

Explain why, under the present conditions, it may be an advantage with many participants in a ROSCA.

c. We will now remove the somewhat unrealistic assumption that all participants keep their promises given at the start of a ROSCA. Assume instead that there is a positive probability, *p*, that someone who wins the pot immediately afterwards drops out of the ROSCA and therefore does not pay the contribution s/n on any of the remaining meetings. Suppose you win the pot in period *v*. Let *X* be the number of participants who have dropped out before you won the pot. For example, if you win the pot in the first period, $X = 0$. If you win in the second period, $X = 0$ or $X = 1$. If you win in the third period, $X = 0$, 1 or 2, and so on. Assume (in this section only) that *p* is the same for every participant and period.

(**i**) Explain why *X* is binomially distributed with $E(X \mid V = v) = (v-1)p$ when $V = v$ is given (fixed), and the participants take their decisions independently of each other.

(**ii**) Show that the marginal expectation of *X* is

$$E(X) = p\frac{n-1}{2}$$

(**iii**) Find an expression for the marginal variance, var(*X*), expressed by *n* and *p*.

d. Suppose that the sanction against someone who drops out mainly consists of exclusion from future participation in the ROSCA, and exclusion from other ROSCA's as well (the ROSCA's are familiar with each other and with the records of potential participants). Let τ denote the welfare loss per year as a result of exclusion, expressed as an amount of money units. By integrating the discounted welfare loss up to the horizon *T*, we obtain the total discounted welfare loss caused by exclusion, *W*(*T*), as

$$W(T) = \int_0^T \tau e^{-\theta t} dt = \frac{\tau}{\theta}(1 - e^{-\theta T})$$

where θ is the discount rate. Suppose further that a participant believes that her remaining lifetime, *T* (measured in years) in the slum is gamma distributed with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$. The parameters α, λ may depend on the person in question but not on *n*. Show that the expected total discounted welfare loss, ω, is

$$\omega = E(W(T)) = \frac{\tau}{\theta}\left(1 - \left(\frac{\lambda}{\lambda + \theta}\right)^{\alpha}\right)$$

e. Whether participants drop out or not after having won the pot will naturally depend on the size of *n*. Hence the probability *p* should depend on *n* as well. With fewer participants the social relationships that help keeping the participants in the ROSCA are stronger. Let us assume that it is only by the threat of exclusion that this control works.

As a "rational" but "heartless" agent you consider dropping out when you win the pot in period *v* according to the following rule:

If the remaining amount, Q, to be paid to the ROSCA until the end of the year after the winning period v, exceeds the expected total discounted welfare loss by exclusion, ω*, you decide to drop out.*

What does this rule mean in terms of the corresponding probability *p* of dropping out?

[**Hint:** Set up an expression for *Q*. You can ignore any discounting in *Q* since the payment period is so short.]

Suppose that the loss per year caused by exclusion, τ, depends on *n* in such a way that it decreases when *n* increases. This assumption appears reasonable since, if the ROSCA's were generally larger, it would be easier to "sneak" back into a ROSCA at a later year, and the exclusion would therefore be less efficient as a sanction. Suppose that all participants are rational and heartless in the sense of adopting the decision rule for dropping out given above. Discuss how *p* varies (i.e., increases or decreases) as a function of *v* and *n*. It is clear that in order to be able to start a ROSCA at all, the individual *p*'s should be close to zero, even for the one that wins after first period. Does this explain why the ROSCA's in actual practice usually are rather small? (Around $n = 13$ is a common size.)

[**Extra hint added to section e:**

The remaining amount to be paid is $Q = \frac{(n-v)s}{n} = \left(1 - \frac{v}{n}\right)s$, where v/n is the time of the year that the winning occurs. The heartless one has $p = 1$ if $Q = \left(1 - \frac{v}{n}\right)s > \omega$ and $p = 0$ if $Q \le \omega$. Explain why ω decreases when *n* increases. Consider what happens to *p* (does it increase or decrease?) in the following two scenarios:

(i) Fix *v* to one of the earlier periods (to $v = 1$ say), and let *n* increase. Explain why *p* increases with *n* so that the ROSCA is feasible only when *n* is sufficiently small.

(ii) Fix the time of the year, (v/n), when a participant wins the pot, and let *n* increase. Explain why *p* then increases, thus again leading to a smaller *n*. **End of hint.**]