K.S, A.S, November 03

Mathematics – ECON4140/4145

Solutions to Problem set 7

Problem 97

See the answer in the booklet.

Problem 104

(a) Since x and y are positive, an equilibrium point must satisfy the equations

$$y - \frac{x}{2} - 2 = 0, \quad 1 - \frac{y}{2x} = 0.$$

It is easy to see that this system has a unique solution, $x = 4/3$, $y = 8/3$. (The second equation gives $y = 2x$, etc.) Thus $(x_0, y_0) = (4/3,\, 8/3)$.

(b) With

$$f(x,y) = x\Big(y - \frac{x}{2} - 2\Big) = xy - \frac{x^2}{2} - 2x$$
$$g(x,y) = y\Big(1 - \frac{y}{2x}\Big) = y - \frac{y^2}{2x}$$

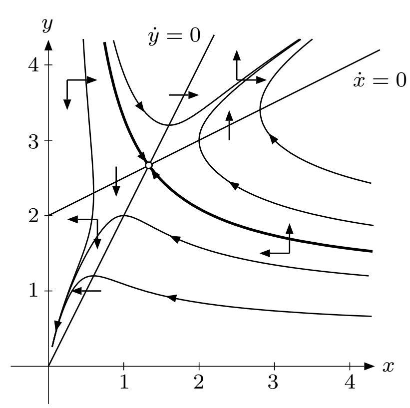
we get

$$\mathbf{A}(x,y) = \begin{pmatrix} f_1'(x,y) & f_2'(x,y) \\ g_1'(x,y) & g_2'(x,y) \end{pmatrix} = \begin{pmatrix} y - x - 2 & x \\ y^2/(2x^2) & 1 - y/x \end{pmatrix}.$$

In particular,

$$\mathbf{A}(x_0,y_0) = \mathbf{A}(4/3,\, 8/3) = \begin{pmatrix} -2/3 & 4/3 \\ 2 & -1 \end{pmatrix},$$

and since $\det(\mathbf{A}(x_0,y_0)) = -2 < 0$, the point (x_0, y_0) is a (local) saddle point.



Problem 104. Phase diagram with some integral curves.

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(c) The zeroclines $\dot{x} = 0$ and $\dot{y} = 0$ are the straight lines $y = x/2 + 2$ and $y = 2x$, respectively. These lines partition the first quadrant into four regions. The signs of $\dot{x}$ and $\dot{y}$ in each of these regions are indicated by arrows in the figure. In the figure we have also drawn some (computer generated) integral curves for the system. The integral curves converging towards the saddle point are drawn thicker than the other curves.

Problem 106

(a) The Hamiltonian for this problem is

$$H(t, x, u, p) = x - u^2 + p(x + u) = (1 + p)x - u^2 + pu.$$

Assume that (x^*, u^*) is an optimal pair. The maximum principle gives us the following three necessary conditions:

(I) For each t, $u^*(t)$ must be a value of u that maximizes

$$H(t, x^*(t), u, p(t)) = (1 + p(t))x^*(t) - u^2 + p(t)u$$

for $u \in (-\infty, \infty)$. Since the control region is open, it follows that we must have

$$(1) \qquad H'_u(t, x^*(t), u^*(t), p(t)) = 0, \quad \text{i.e.} \quad -2u^*(t) + p(t) = 0.$$

(II) For each t, $\dot{p} = -H'_x((t, x^*(t), u^*(t), p(t))$, i.e.

$$(2) \qquad \dot{p}(t) = -1 - p(t).$$

(III) Since $x(2)$ is free, we have

$$(3) \qquad p(2) = 0.$$

Of course, we must also have

$$(4) \qquad \dot{x}^* - x^* = u^*, \qquad x^*(0) = 0.$$

The Hamiltonian is concave with respect to (x, u). Hence, if we can find functions x^*, u^*, and p that satisfy the conditions (1)–(4), we know that they will form an optimal solution.

The differential equation (2) has the general solution $p(t) = Ce^{-t} - 1$. Since $p(2)$ must be 0, it follows that $C = e^2$, and so

$$p(t) = e^{2-t} - 1.$$

It follows from (1) that

$$u^*(t) = \frac{1}{2}p(t) = \frac{1}{2}(e^{2-t} - 1).$$

Equation (4) now becomes

$$(5) \qquad \dot{x}^* - x^* = u^*(t) = \frac{1}{2}e^{2-t} - \frac{1}{2}.$$

The homogeneous equation $\dot{x} = x$ has the general solution $x = Ae^t$, so the general solution of (5) is $x^* = Ae^t + w^*$, where $w^* = w^*(t)$ is a particular solution of (5). We try with a w^* of the form $w^* = Be^{2-t} + D$. Then

$$\dot{w}^* - w^* = -2Be^{2-t} - D = \frac{1}{2}e^{2-t} - \frac{1}{2} \iff B = -\frac{1}{4},\ D = \frac{1}{2}\,.$$

Hence $x^*(t) = Ae^t - \frac{1}{4}e^{2-t} + \frac{1}{2}$ for a suitable constant A. (We could here also use the formula for the solution of a linear differential equation.) The initial condition $x^*(0) = 0$ gives

$$A - \frac{1}{4}e^2 + \frac{1}{2} = 0 \iff A = \frac{1}{4}e^2 - \frac{1}{2}\,,$$

and so

$$x^*(t) = \Big(\frac{1}{4}e^2 - \frac{1}{2}\Big)e^t - \frac{1}{4}e^{2-t} + \frac{1}{2} = \frac{1}{4}(e^{2+t} - e^{2-t}) - \frac{1}{2}(e^t - 1).$$

(b) As in part (b) we have

$$\dot{p}(t) = -1 - p(t), \quad p(2) = 0,$$

and so we still have $p(t) = e^{2-t} - 1$. However, the maximization of the Hamiltonian with respect to u becomes a bit more complicated than in (1), since u is now confined to the closed interval $[0, 1]$. Considered as a function of u, the Hamiltonian

$$H(t, x, p, u) = (1 + p)x - u^2 + pu$$

is a quadratic polynomial. The graph of this polynomial is a parabola whose highest point corresponds to $u = p/2$. Therefore

$$u^*(t) = \begin{cases} 1 & \text{if } p(t) > 2, \\ p(t)/2 & \text{if } 0 \le p(t) \le 2, \\ 0 & \text{if } p(t) < 0. \end{cases}$$

Since $\dot{p} < 0$, p is strictly decreasing. Moreover, $p(0) = e^2 - 1 > 1$ and $p(2) = 0 < 1$. Hence there exists a unique t^* between 0 and 2 with $p(t^*) = 2$. We have

$$p(t^*) = 2 \iff e^{2-t^*} - 1 = 2 \iff 2 - t^* = \ln 3 \iff t^* = 2 - \ln 3 \approx 0.9014.$$

It follows that

$$u^*(t) = \begin{cases} 1 & \text{if } 0 \le t \le t^*, \\ \frac{1}{2}p(t) = \frac{1}{2}e^{2-t} - \frac{1}{2} & \text{if } t^* < t \le 2. \end{cases}$$

For t in $(0, t^*)$ we then have $\dot{x}^*(t) = x^*(t) + 1$, which gives $x^*(t) = Ke^t - 1$ for some constant K. Since $x^*(t) = 0$, we have $K = 1$ and $x^*(t) = e^t - 1$.

For $t > t^*$, we have

$$\dot{x}^*(t) - x^*(t) = u^*(t) = \frac{1}{2}e^{2-t} - \frac{1}{2}\,.$$

This is precisely equation (5) above, and we saw that the general solution of this equation is

$$x^*(t) = A_1e^t - \frac{1}{4}e^{2-t} + \frac{1}{2}\,.$$

It only remains to determine the value of the constant A_1. Since x^* is a continuous function, the one-sided limits of $x^*(t)$ as t tends to t^* must be equal, that is,

$$\lim_{t\to(t^*)^-}(e^t-1)=\lim_{t\to(t^*)^+}\Big(A_1e^t-\frac{1}{4}e^{2-t}+\frac{1}{2}\Big),$$
$$e^{t^*}-1=A_1e^{t^*}-\frac{1}{4}e^{2-t^*}+\frac{1}{2}\,,$$

and finally

$$A_1=1+\frac{1}{4}e^{2-2t^*}-\frac{3}{2}e^{-t^*}=1+\frac{9}{4}e^{-2}-\frac{9}{2}e^{-2}=1-\frac{9}{4}e^{-2}.$$

Therefore

$$x^*(t)=\begin{cases}e^t-1 & \text{if } 0\le t\le t^*,\\ A_1e^t-\frac{1}{4}e^{2-t}+\frac{1}{2}=e^t-\frac{1}{4}e^{2-t}-\frac{9}{4}e^{t-2}+\frac{1}{2} & \text{if } t^*\le t\le 2.\end{cases}$$

Problem 108

(a) With $F(t,x,\dot x)=(-2\dot x-\dot x^2)e^{-t/10}$ we have

$$\frac{\partial F}{\partial x}=0\qquad\text{and}\qquad\frac{\partial F}{\partial \dot x}=(-2-2\dot x)e^{-t/10}.$$

The Euler equation is therefore

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\partial F}{\partial \dot x}\Big)=0\iff -2\ddot x e^{-t/10}+(-2-2\dot x)e^{-t/10}\Big(-\frac{1}{10}\Big)=0\\ (*)\qquad &\qquad\qquad\iff \ddot x-\frac{1}{10}\dot x=\frac{1}{10}\end{aligned}$$

This is a second-order differential equation with constant coefficients. The characteristic equation of the corresponding homogeneous equation is $r^2-r/10=0$, which has the roots $r_1=1/10$ and $r_2=0$. Thus, $(*)$ has the general solution $x=Ae^{t/10}+B+u^*$, where u^* is some particular solution of $(*)$.

To find u^* we try with a function of the form $u^*=Ct+D$. This gives $\dot u^*=C$, $\ddot u^*=0$, and so we see that $u^*=Ct+D$ is a solution of $(*)$ if and only if $C=-1$. We choose $u^*=-t$, and it follows that the general solution of $(*)$ is $x(t)=Ae^{t/10}+B-t$.

A solution that satisfies the boundary conditions of the problem must be such that

$$x(0)=A+B=1\qquad\text{and}\qquad x(1)=Ae^{1/10}+B-1=0.$$

This gives

$$A=0,\quad B=1,$$

and finally

$$x=1-t.$$

This is the only solution of the Euler equation that satisfies the boundary conditions. It is optimal because $F(t,x,\dot x)$ is concave with respect to $(x,\dot x)$.

Note: Another way to solve the Euler equation is to note that $\dfrac{\partial F}{\partial \dot{x}}$ must be constant because $\dfrac{d}{dt}\Big(\dfrac{\partial F}{\partial \dot{x}}\Big) = 0$. Thus, for some constant C,

$$\frac{\partial F}{\partial \dot{x}} = C \iff (-2 - 2\dot{x})e^{-t/10} = C \iff \dot{x} = -1 - \tfrac{1}{2}Ce^{t/10}.$$

Integration then gives

$$x = -t - 5Ce^{t/10} + D$$

as the general solution of the Euler equation. To find the solution that satisfies the boundary conditions, we can determine the constants C and D from $x(0) = 1$ and $x(1) = 0$ as before.

(b) We now have

$$F(t, x, \dot{x}) = (-\dot{x}ae^{\alpha t} - \dot{x}^2 e^{\beta t})e^{-rt} = -a\dot{x}e^{(\alpha - r)t} - \dot{x}^2 e^{(\beta - r)t}.$$

This gives

$$\frac{\partial F}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial \dot{x}} = -ae^{(\alpha - r)t} - 2\dot{x}e^{(\beta - r)t}.$$

The Euler equation becomes

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial F}{\partial \dot{x}}\Big) = 0 &\iff -a(\alpha - r)e^{(\alpha - r)t} - 2\ddot{x}e^{(\beta - r)t} - 2\dot{x}(\beta - r)e^{(\beta - r)t} = 0 \\
&\iff \big(2\ddot{x} - 2(\beta - r)\dot{x}\big)e^{(\beta - r)t} = -a(\alpha - r)e^{(\alpha - r)t} \\
&\iff \ddot{x} - (\beta - r)\dot{x} = -\tfrac{1}{2}a(\alpha - r)e^{(\alpha - \beta)t}.
\end{aligned}$$

Since $F(t, x, \dot{x}) = e^{-rt}\pi$, where π is the profit function, the Euler equation

$$\frac{d}{dt}\Big(\frac{\partial F}{\partial \dot{x}}\Big) = 0$$

tells us that

$$\frac{\partial F}{\partial \dot{x}} = e^{-rt}\frac{\partial \pi}{\partial \dot{x}}$$

must be constant, i.e. for some constant c,

$$e^{-rt}\frac{\partial \pi}{\partial \dot{x}} = c \iff \frac{\partial \pi}{\partial \dot{x}} = ce^{rt}.$$

Problem 114

Using the Lagrangian

$$\mathcal{L}(x, y, z) = 100 - e^{-x} - e^{-y} - e^{-z} - \lambda(x + y + z - a) - \mu(x - b),$$

we get the first-order conditions

(1) $\quad \mathcal{L}_1'(x, y, z) = e^{-x} - \lambda - \mu = 0$

(2) $\quad \mathcal{L}_2'(x, y, z) = e^{-y} - \lambda = 0$

(3) $\quad \mathcal{L}_3'(x, y, z) = e^{-z} - \lambda = 0$

(4) $\quad \lambda \geq 0, \qquad \lambda = 0 \text{ if } x + y + z < a$

(5) $\quad \mu \geq 0, \qquad \mu = 0 \text{ if } x < b$

It is easy to see that $-e^{-x}$ is a concave function of x, and in fact a concave function of all the three variables. Similarly, e^{-y} and e^{-z} are concave functions. Furthermore, $x+y+z$ and x are linear, and hence concave. Therefore, $\mathcal{L}$ is concave as a sum of concave functions.

(b) From (2), $\lambda = e^{-y} > 0$, and so (4) implies

$$x + y + z = a. \tag{6}$$

Further, (2) and (3) show that $e^{-y} = e^{-z}$, so

$$y = z. \tag{7}$$

From the constraints in the problem we have $x \leq b$.

A. <u>Suppose $x = b$</u>. Then (6) and (7) give $b + 2z = a$, so $y = z = \frac{1}{2}(a - b)$. We know that $\lambda = e^{-y} > 0$, but it remains to check the sign of μ. From (1),

$$\mu = e^{-x} - \lambda = e^{-x} - e^{-y} \geq 0 \iff x \leq y \iff b \leq \tfrac{1}{2}(a-b) \iff a \geq 3b.$$

So for $a \geq 3b$, then, the point

$$(x, y, z) = (b, \tfrac{1}{2}(a-b), \tfrac{1}{2}(a-b))$$

satisfies all the Kuhn–Tucker conditions with $\lambda = e^{-(a-b)/2}$ and $\mu = e^{-b} - e^{-(a-b)/2}$.

B. <u>Now suppose $x < b$</u>. Then (5) implies $\mu = 0$, and equations (1)–(3) imply that $e^{-x} = e^{-y} = e^{-z}$, so $x = y = z$. From (6),

$$x = y = z = \tfrac{1}{3}a,$$

and we have $\lambda = e^{-a/3}$, $\mu = 0$. Note that in this case $x < b$, so $\frac{1}{3}a < b$, i.e. $a < 3b$.

Since $\mathcal{L}$ is concave, we have found the solution.

(c) For $a \geq 3b$, the value function is

$$f^*(a, b) = 100 - e^{-b} - 2e^{-(a-b)/2},$$

and

$$\frac{\partial f^*}{\partial a} = e^{-(a-b)/2} = \lambda, \qquad \frac{\partial f^*}{\partial b} = e^{-b} - e^{-(a-b)/2} = \mu$$

(at least if $a > 3b$). For $a < 3b$,

$$f^*(a, b) = 100 - 3e^{-a/3}, \qquad \frac{\partial f^*}{\partial a} = e^{-a/3} = \lambda, \qquad \frac{\partial f^*}{\partial b} = 0 = \mu.$$

(d) We have

$$F^*(a) = f^*(a, 0) = \begin{cases} 99 - 2e^{-a/2} & \text{if } a \geq 0, \\ 100 - 3e^{-a/3} & \text{if } a < 0, \end{cases}$$

and it is clear that F^* is continuous at 0 (with $F^*(0) = 97$).

The derivative of F^* is

$$(F^*)'(a) = \begin{cases} e^{-a/2} & \text{if } a > 0, \\ e^{-a/3} & \text{if } a < 0. \end{cases}$$

Moreover, the two one-sided derivatives of $F^*(a)$ at $a = 0$ exist and are equal to $e^0 = 1$. It follows that $(F^*)'(a)$ is a decreasing function of a, and so F^* is concave.

Problem 85

(a) The Hamiltonian is

$$H(t, x, u, p) = (x - u) + p(aue^{-2t} - x) = (1 - p)x + (ape^{-2t} - p_0)u.$$

If (x^*, u^*) solves the problem, then there exists a continuous function p (defined on $[0, T]$) so that

$$\text{(i)} \qquad \dot{p}(t) = -\frac{\partial H^*}{\partial x}(t) = p - 1.$$

For each t in $[0, T]$, $u = u^*(t)$ is that value of u in $[0, 1]$ which maximizes

$$H(t, x^*(t), u, p(t)) = (1 - p(t))x^*(t) + (ap(t)e^{-2t} - 1)u$$

Thus,

$$\text{(ii)} \qquad u^*(t) = \begin{cases} 1 & \text{if } ap(t) > e^{2t}, \\ 0 & \text{if } ap(t) < e^{2t}. \end{cases}$$

Moreover,

$$\text{(iii)} \qquad p(T) = 0$$

since $x(T)$ is free.

The general solution of (i) is $p(t) = Ce^t + 1$, where C is an arbitrary constant. Condition (iii) gives $C = -e^{-T}$, so that

$$p(t) = 1 - e^{t-T}.$$

Define $\varphi(t) = ap(t) - e^{2t}$. From (ii) we see that

$$u^*(t) = \begin{cases} 1 & \text{if } \varphi(t) > 0, \\ 0 & \text{if } \varphi(t) < 0. \end{cases}$$

The function φ is strictly decreasing, so φ has at most one zero in $[0, T]$. ($\dot{\varphi} = a\dot{p} - 2e^{2t} = -ae^{t-T} - 2e^{2t} < 0$.)

Since $p(T) = 0$, $\varphi(T) < 0$. In (b) and (c) we need to observe that there are two possibilities:

(I) If $\varphi(0) > 0$, then φ has a unique zero t^* in $(0, T)$ and

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, t^*], \\ 0 & \text{if } t \in (t^*, T]. \end{cases}$$

(u^* is left continuous at t^*.)

(II) If $\varphi(0) \le 0$, then $\varphi(t) < 0$ for all $t > 0$, and then $u^*(t) = 0$ for all t i $[0, T]$.

(b) With $T = \ln 10$, $a = 5$ and $x_0 = 5$ we get

$$\varphi(t) = 5 - 5e^{t-\ln 10} - e^{2t} = 5 - \tfrac{1}{2}e^t - e^{2t}.$$

Then $\varphi(0) > 0$, and we have the situation described in (I) above. The zero t^* for φ is determined by the equation

$$e^{2t^*} + \tfrac{1}{2}e^{t^*} - 5 = 0.$$

This is a second degree equation in e^{t^*} and we get

$$e^{t^*} = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + 20}}{2} = \frac{-\frac{1}{2} \pm \frac{9}{2}}{2} = \frac{-1 \pm 9}{4} = \begin{cases} 2 \\ -5/2 \end{cases}$$

where only the positive solution makes sense. Then $t^* = \ln 2$ and

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, \ln 2], \\ 0 & \text{if } t \in (\ln 2, \ln 10]. \end{cases}$$

For $t < \ln 2$, $\dot{x}^* = 5e^{-2t} - x^*$, which gives $x^*(t) = C_1 e^{-t} - 5e^{-2t}$. Since $x^*(0) = 5$, we get $C_1 = 10$ and

$$x^*(t) = 10e^{-t} - 5e^{-2t}, \qquad t \le \ln 2.$$

In particular, $x^*(\ln 2) = 10e^{-\ln 2} - 5^{-2\ln 2} = 10/2 - 5/4 = 15/4$.

For $t > \ln 2$, $u^*(t) = 0$ and $\dot{x}^* = -x^*$, which gives

$$x^*(t) = x(\ln 2)e^{\ln 2 - t} = \frac{15}{4} \cdot 2e^{-t} = \frac{15}{2}e^{-t}, \qquad t \ge \ln 2.$$

(c) With $T = \ln 10$, $a = 1/2$ and $x_0 = 5$, we get $\varphi(0) = a - ae^{-T} - 1 < 0$, so

$$u^*(t) \equiv 0.$$

Hence $\dot{x}^* = -x^*$, which gives

$$x^*(t) = x^*(0)e^{-t} = 5e^{-t}.$$