University of Oslo
Department of Economics

ECON4140/ECON4145 Mathematics 3

Tuesday 7 December 2004, 09.00–12.00.

There are 2 pages of problems to be solved.

All printed and written material may be used. Pocket calculators are allowed.

State reasons for all your answers.

Grades given: A (best), B, C, D, E and F, with E as the weakest passing grade.

Problem 1

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= -x^2 + 8x - 2y \end{aligned}$$

(a) Find the equilibrium points and classify them (if possible) as locally asymptotically stable equilibrium points or saddle points.

(b) Draw a phase diagram for the system in the first quadrant ($x > 0$, $y > 0$), and indicate some possible integral curves.

Problem 2

(a) Let $g(x, y) = x^{-p} + y^{-q}$, $x > 0$, $y > 0$. For which values of p and q is g concave?

(b) Where in the xy-plane is $f(x, y) = 2x + y - \frac{1}{3}x^3 - xy - y^2$ concave?

(c) Consider the nonlinear programming problem

$$\text{maximize } 2x + y - \tfrac{1}{3}x^3 - xy - y^2 \quad \text{subject to} \quad x \geq \tfrac{1}{4}, \quad x + y \leq 3$$

Write down the Kuhn–Tucker condidions. Why are they sufficient for optimality in this problem?

(d) Solve the problem in (c).

(Cont.)

Problem 3

(a) Solve the control problem (T fixed positive number)

$$\max \int_0^T [100 - \frac{1}{2}(x - 2u)^2]\, e^{0.5t}\, dt, \quad \dot{x} = 4u - x,\ x(0) = 0,\ x(T) = 10,\ u \in \mathbb{R}$$

(b) Find the solution if the initial and terminal conditions are $x(0) = x_0$ and $x(T)$ is free. (T fixed positive number.)

Problem 4

(Difficult. Can be skipped.)
Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices that have the same eigenvalues λ_1 and λ_2, and the same corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. Show that $\mathbf{B} = \mathbf{A}$.

(Cont.)