University of Oslo
Department of Economics

ECON4140/4145 Mathematics 3

Friday 8 December 2006, 09.00–12.00.

There are 2 pages of problems to be solved.

All printed and written material may be used. Pocket calculators are allowed.

State reasons for all your answers.

Grades given: A (best), B, C, D, F, with D as the weakest passing grade.

Problem 1

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= y - e^{2-x} - 1 \\ \dot{y} &= x^2 - xy \end{aligned}$$

(a) Show that $(2, 2)$ is an equilibrium point. Find all equilibrium points.

(b) Classify the equilibrium point $(2, 2)$, if possible.

(c) Draw a phase diagram in the first quadrant (i.e. for $x \geq 0$, $y \geq 0$) and indicate some possible integral curves.

(d) Compute the eigenvalues of the Jacobian at $(2, 2)$. Find an eigenvector associated with each eigenvalue.

Problem 2

Consider the nonlinear programming problem

$$\text{maximize} \quad x + 3y - 4e^{-x-y} \quad \text{subject to} \quad \begin{cases} x + 2y \leq 2 \\ x + y \leq 1 \end{cases}$$

(a) Write down the necessary Kuhn–Tucker conditions for a point (x, y) to be a solution of the problem. Are the conditions also sufficient?

(b) Solve the problem.

(Cont.)

Problem 3

Consider the optimal control problem

$$\max \int_0^T (2x - 6u)e^{-t}\,dt, \quad \dot{x} = x + u, \;\; x(0) = 0, \;\; x(T) \text{ free}, \;\; u \in [0, 1]$$

assuming that $T > 3$.

(a) Write down the conditions in the maximum principle for a pair $(x^*(t), u^*(t))$ to solve the problem. Are the conditions sufficient for optimality in this case?

(b) Solve the problem.

(c) Let $V(T) = \int_0^T (2x^*(t) - 6u^*(t))e^{-t}\,dt$ and let $H^*(t)$ denote the Hamiltonian evaluated along the optimal solution. Verify that $V'(T) = H^*(T)$.