ECON4140/ECON4145, Mathematics 3

Wednesday 10 December 2003, 14.30–17.30.

There are 2 pages of problems to be solved.

All printed and written material may be used. Pocket calculators are allowed.

State reasons for all your answers.

Grades given: A (best), B, C, D, E and F, with E as the weakest passing grade.

Problem 1

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= ax - by^{b-1} \\ \dot{y} &= x - cy \end{aligned} \qquad x > 0,\ y > 0$$

where a, b and c are positive constants, with $b < 2$.

(a) Find the equilibrium point and classify it (if possible) as a locally asymptotically stable equilibrium point or a saddle point.

(b) If $a = 1/32$, $b = 1/2$ and $c = 2$ the equilibrium point is $(x^*, y^*) = (8, 4)$. Draw a phase diagram for this case and indicate some possible integral curves.

Problem 2

(a) Solve the differential equation

$$\ddot{x} + 4\dot{x} + 9x = t + 1$$

(b) In a labour market model one encounters the differential equation

$$F'(w) - \frac{1}{1-w}F(w) = \frac{a}{1-w} \qquad (*)$$

where $w \in [0, 1)$, and where $F(w)$ is the unknown function, a is a constant. Find the general solution of this equation.

(c) Verify that the solution of $(*)$ which has $F(0) = 0$ is $F^*(w) = aw/(1-w)$.

(Cont.)

Problem 3

(a) Reformulate the problem

$$\text{minimize } 4\ln(x^2+2)+y^2 \quad \text{subject to} \quad x^2+y\geq 2, \quad x\geq 1$$

as a standard Kuhn–Tucker maximization problem and write down the necessary Kuhn–Tucker conditions.

(b) Find the solution to the problem. (You may take it for granted that there is a solution.)

Problem 4

Consider the control problem

$$\max \int_0^1 \sqrt{x-u-1}\,dt,\ \dot{x}=u,\ x(0)=0,\ x(1)=x_1,\ u\in(-\infty,\infty)$$

(a) Write down the conditions in the maximum principle.

(b) Find the only possible solution $(x^*(t),u^*(t))$ to the necessary conditions when $x_1=1-e^2$, and prove that it is optimal. Verify that $x^*(t)-u^*(t)-1>0$ for all t in $[0,1]$.

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