ECON4140/ECON4145 Mathematics 3

Friday 9 December 2005, 09.00–12.00.

There are 2 pages of problems to be solved.

All printed and written material may be used. Pocket calculators are allowed.

State reasons for all your answers.

Grades given: A (best), B, C, D, F, with D as the weakest passing grade.

Problem 1

Consider the system of differential equations

$$\dot{x} = -\tfrac{1}{4}x^2 - y + 6$$
$$\dot{y} = -x^2 + 4y^2$$

(a) Find all equilibrium points.

(b) Prove that the equlibrium point in the first quadrant is a local saddle point.

(c) Draw a phase diagram in the first quadrant (for $x \geq 0$, $y \geq 0$) and indicate some possible integral curves.

Problem 2

Consider the nonlinear programming problem

$$\text{maximize} \quad xy \quad \text{subject to} \quad \begin{cases} x^2 + ry^2 \leq m \\ x \geq 1 \end{cases}$$

Here r and m are positive constants, $m > 1$.

(a) Write down the necessary Kuhn–Tucker conditions for a point (x^*, y^*) to be a solution of the problem.

(b) Solve the problem.

(Cont.)

Problem 3

Find the Euler equation for the variational problem:

$$\max \int_{t_0}^{t_1} (tx - x^2 - t^2\dot{x} - \tfrac{1}{2}\dot{x}^2)e^{-t}\,dt, \quad x(t_0) = x_0, \quad x(t_1) = x_1$$

(You are not supposed to find the solution of the Euler equation.)

Problem 4

Consider the optimal control problem

$$\max \int_0^1 (x - e^{-u})\,dt, \quad \dot{x} = -u,\ \ x(0) = 0,\ \ x(1) \text{ free},\ \ u \leq 1$$

(a) Write down the conditions in the maximum principle for a pair $(x^*(t), u^*(t))$ to solve the problem. Verify that the conditions in this case are sufficient for optimality.

(b) Solve the problem.