Seminar 3 - ECON 4140

Problem 2 exam 05

max xy given $\begin{cases} x^2 + ry^2 \le m \\ x \ge 1 \end{cases}$, $r > 0, m > 1$

a) $\mathcal{L}(x,y) = xy - \lambda(x^2 + ry^2 - m) - \mu(1 - x)$

$$\boxed{\begin{array}{ll} (1) & \frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda x + \mu = 0 \\ (2) & \frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda r y = 0 \\ (3) & \lambda \ge 0 \ (\lambda = 0 \text{ if } x^2 + ry^2 < m) \\ (4) & \mu \ge 0 \ (\mu = 0 \text{ if } x > 1) \\ (5) & x^2 + ry^2 \le m \\ (6) & x \ge 1 \end{array}}$$

b) • Guess (5) and (6) binding (equality)

$\Rightarrow \boxed{x = 1}$ $y^2 = \frac{m-1}{r}$ $\underline{y = \pm\sqrt{\frac{m-1}{r}}}$ (from (5) and (6))

(2) gives $1 = 2\lambda r y$ that is $\lambda = \frac{1}{2yr}$ since $\lambda \ge 0$ $\underline{y = \sqrt{\frac{m-1}{r}}}$

$\Rightarrow \lambda = \frac{1}{2r} \frac{\sqrt{r}}{\sqrt{m-1}} = \underline{\frac{1}{2\sqrt{r}\sqrt{m-1}}}$

(1) gives $\mu = \frac{1}{\sqrt{r}\sqrt{m-1}} - \frac{\sqrt{m-1}}{\sqrt{r}} = \frac{1}{\sqrt{r}\sqrt{m-1}} - \frac{m-1}{\sqrt{r}\sqrt{m-1}} = \underline{\frac{2-m}{\sqrt{r}\sqrt{m-1}}}$ ok if $m \le 2$ ($\mu \ge 0$)

Anyway, this gives $\boxed{xy = \sqrt{\frac{m-1}{r}}}$

• Guess (5) binding and (6) non-binding

$\Rightarrow x^2 + ry^2 = m$, $x > 1$ $\Rightarrow \underline{\mu = 0}$ (1) gives $\underline{\lambda = \frac{y}{2x}}$

(2) then gives $x = 2ry \cdot \frac{y}{2x}$ $\Rightarrow \underline{x^2 = ry^2}$

(5) then gives $2ry^2 = m \Rightarrow y^2 = \frac{m}{2r}$ $\quad \underline{y = \pm\sqrt{\frac{m}{2r}}}$

$\Rightarrow x^2 = r\frac{m}{2r} = \frac{m}{2} \Rightarrow$ $\underline{x = \pm\sqrt{\frac{m}{2}}}$

Since $\lambda = \frac{y}{2x} \geq 0$, either we have $\underline{(x,y) = \left(\sqrt{\frac{m}{2}}, \sqrt{\frac{m}{2r}}\right)}$ or $\underline{(x,y) = \left(-\sqrt{\frac{m}{2}}, -\sqrt{\frac{m}{2r}}\right)}$

Both choices give $\boxed{xy = \frac{m}{2\sqrt{r}}}$

There is no reason to check unbinding (5) by logical reasoning (want the combination of x^2 and y^2 as large as possible to maximize xy).

$\Rightarrow$ Need to compare $xy = \sqrt{\frac{m-1}{r}}$ and $xy = \frac{m}{2\sqrt{r}}$

First is best if $\sqrt{\frac{m-1}{r}} \geq \frac{m}{2\sqrt{r}} \Rightarrow \sqrt{m-1} \geq \frac{m}{2}$

$m - 1 \geq \frac{m^2}{4}$ $\quad \underline{m^2 - 4m + 4 \leq 0}$ that is $(m-2)^2 \leq 0$ Impossible

but equality possible

$\Rightarrow$ $\boxed{\begin{array}{l} \underline{(x,y) = \left(\sqrt{\frac{m}{2}}, \sqrt{\frac{m}{2r}}\right) \text{ and } (x,y) = \left(-\sqrt{\frac{m}{2}}, -\sqrt{\frac{m}{2r}}\right) \text{ is optimal}} \\ \text{(In the special case } m=2,\ (x,y) = \left(1, \sqrt{\frac{m-1}{r}}\right) \text{ is also optimal).} \end{array}}$

Problem 3 exam 07

$$\max\ -(x-6)^2-(y-5)^2 \quad \text{s.t.} \begin{cases} x^2+y^2 \le 25 \\ a(x-3)+y \le 4 \end{cases} \qquad a \ne \frac{3}{4}$$

a) $\mathcal{L} = -(x-6)^2-(y-5)^2-\lambda(x^2+y^2-25)-\mu(a(x-3)+y-4)$

(1) $\frac{\partial \mathcal{L}}{\partial x} = -2(x-6)-2\lambda x-\mu a = 0$

(2) $\frac{\partial \mathcal{L}}{\partial y} = -2(y-5)-2\lambda y-\mu = 0$

(3) $\lambda \ge 0$ ($\lambda = 0$ if $x^2+y^2<25$)

(4) $\mu \ge 0$ ($\mu = 0$ if $a(x-3)+y<4$)

(5) $x^2+y^2 \le 25$

(6) $a(x-3)+y \le 4$

b) ~~when~~ For what a will $(x,y)=(3,4)$ be optimal?

If $x=3$ and $y=4$, then (5) gives $x^2+y^2 = 9+16 = 25$, that is (5) is binding

And (6) gives $a \cdot 0 + 4 = 4$, that is (6) is binding.

$\Rightarrow$ No more information about λ and μ yet.

(1) gives $6-6\lambda-\mu a = 0 \Rightarrow \boxed{\mu = \frac{6(1-\lambda)}{a}}$

(2) then gives $2-8\lambda = \frac{6(1-\lambda)}{a}$ $\quad 2a-8a\lambda = 6-6\lambda \quad 2(a-3) = 2\lambda(4a-3)$

that is, $\boxed{\lambda = \frac{a-3}{4a-3}}$ $\Rightarrow$ $\mu = \frac{6\left(\frac{4a-3}{4a-3}-\frac{a-3}{4a-3}\right)}{a} = \frac{6\left(\frac{3a}{4a-3}\right)}{a} = \frac{18}{4a-3}$

$\boxed{\mu = \frac{18}{4a-3}}$ Need $\mu \ge 0$ which is ok when $4a-3 > 0$ or $a > \frac{3}{4}$

Need $\lambda \ge 0$ which is ok when $\frac{a-3}{4a-3} \ge 0$ $\quad a-3 \ge 0 \quad a \ge 3$ -

$\Rightarrow$ Optimal when $a \ge 3$

Problem 3-03

a) max $x + xy$ s.t. $y + x^2 e^y \leq 1$

$$\mathcal{L} = x + xy - \lambda(y + x^2 e^y - 1)$$

(1) $\dfrac{\partial \mathcal{L}}{\partial x} = 1 + y - 2\lambda x e^y = 0$

(2) $\dfrac{\partial \mathcal{L}}{\partial y} = x - \lambda(1 + x^2 e^y) = 0$

(3) $\lambda \geq 0$ ($\lambda = 0$ if $y + x^2 e^y < 1$)

(4) $y + x^2 e^y \leq 1$

b) • $(0, -1)$:

(4) gives $-1 < 1$ ok! (4) is satisfied, ($\Rightarrow \lambda = 0$)

(1) gives $0 = 0$ ok!

(2) gives $\lambda = 0$ ok! <u>(1)-(4) all satisfied</u> $\underline{\underline{\lambda = 0}}$

• $(1, 0)$:

(4) gives $1 \leq 1$ ok!

(1) gives $1 = 2\lambda$ $\boxed{\lambda = \frac{1}{2}}$ ok!

(2) gives $\lambda(1+1) = 1$ $\Rightarrow \underline{\lambda = \frac{1}{2}}$ ok!

<u>(1) - (4) all satisfied</u> $\underline{\underline{\lambda = \frac{1}{2}}}$

Problem 3-05

$$\max \ x^2 y e^{-x-y} \quad \text{s.t.} \begin{cases} x \geq 1 \\ y \geq 1 \\ x+y \geq 4 \end{cases}$$

a) $\mathcal{L} = x^2 y e^{-x-y} - \lambda_1(1-x) - \lambda_2(1-y) - \lambda_3(4-x-y)$

$$\boxed{\begin{array}{ll}
(1) & \frac{\partial \mathcal{L}}{\partial x} = 2xye^{-x-y} - x^2ye^{-x-y} + \lambda_1 + \lambda_3 = 0 \\
(2) & \frac{\partial \mathcal{L}}{\partial y} = x^2e^{-x-y} - x^2ye^{-x-y} + \lambda_2 + \lambda_3 = 0 \\
(3) & \lambda_1 \geq 0 \quad (\lambda_1 = 0 \text{ if } x > 1) \\
(4) & \lambda_2 \geq 0 \quad (\lambda_2 = 0 \text{ if } y > 1) \\
(5) & \lambda_3 \geq 0 \quad (\lambda_3 = 0 \text{ if } x+y > 4) \\
(6) & x \geq 1 \\
(7) & y \geq 1 \\
(8) & x+y \geq 4
\end{array}}$$

b) zero binding

- Non binding $\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

$\Rightarrow$ (2) gives $\underline{y=1}$ $\boxed{\text{Impossible}}$

└ Not ok since (7) not binding

1 binding

- (6) binding, (7), (8) nonbinding $\Rightarrow \underline{\lambda_2 = \lambda_3 = 0}$, $\underline{x = 1}$

(2) gives $\underline{y=1}$ (1) gives $\lambda_1 = -e^{-2} < 0$ $\boxed{\text{Impossible}}$

- (7) only binding $\Rightarrow \underline{\lambda_1 = \lambda_3 = 0}$, $\underline{y=1}$

(1) gives $\underline{x=2}$ (2) gives $\underline{\lambda_2 = 0}$

That is $\boxed{(x,y) = (2,1)}$ $\boxed{\text{Impossible}}$ (since $x+y \geq 4$).

- only (8) binding $\Rightarrow$ $\underline{\lambda_1 = \lambda_2 = 0}$ $\underline{x+y=4}$

(1) gives $\lambda_3 = xye^{-4}[x-2]$

(2) gives $x^2 e^{-4}(1-y) = (2-x)xye^{-4} \Rightarrow x(1-y) = (2-x)y$

$\Rightarrow x - xy = 2y - xy$ $\underline{x = 2y}$ $\Rightarrow$ $3y = 4$ $\underline{y = \frac{4}{3}} > 1$

$\Rightarrow x = 4 - \frac{4}{3} = \frac{8}{3}$ $\underline{x = \frac{8}{3}}$ $\Rightarrow \lambda_3 = \frac{32}{9}e^{-4}\left(\frac{2}{3}\right) > 0$ ok.

So $\boxed{(x,y) = \left(\frac{8}{3}, \frac{4}{3}\right) \text{ is candidate}}$

2 binding

- $x=1$ $y=1$ $x+y>4$ $\boxed{\text{impossible}}$

- $\underline{x=1}$ $y>1$ $\underline{x+y=4}$ $\underline{\lambda_2=0}$ $\Rightarrow$ $\underline{y=3}$

(2) gives $\underline{\lambda_3 = 2e^{-4}} > 0$ (1) gives $\lambda_1 = -2e^{-4} - 3e^{-4} < 0$ $\boxed{\text{impossible}}$

- $x>1$ $\underline{y=1}$ $\underline{x+y=4}$ $\underline{\lambda_1=0}$ $\Rightarrow$ $\underline{x=3}$

(1) gives $\lambda_3 = 9e^{-4} - 6e^{-4} = \underline{3e^{-4} > 0}$

(2) gives $\lambda_2 = -3e^{-4}$ $\boxed{\text{impossible}}$

3 binding $x=1$ $y=1$ $x+y=4$ $\boxed{\text{impossible}}$

c) Yes, the only candidate was $(x,y) = \left(\frac{8}{3}; \frac{4}{3}\right)$

Problem 3-12

a) $\max\ -(x+\frac{1}{2})^2-\frac{1}{2}y^2$ s.t. $\begin{cases} y \ge e^{-x} \\ y \le \frac{2}{3} \end{cases}$

$$\mathcal{L}(x,y) = -(x+\tfrac{1}{2})^2 - \tfrac{1}{2}y^2 - \lambda(e^{-x}-y) - \mu(y-\tfrac{2}{3})$$

$$\boxed{\begin{array}{ll}
(1) & \frac{\partial \mathcal{L}}{\partial x} = -2(x+\frac{1}{2}) + \lambda e^{-x} = 0 \\
(2) & \frac{\partial \mathcal{L}}{\partial y} = -y + \lambda - \mu = 0 \\
(3) & \lambda \ge 0 \quad (\lambda = 0 \text{ if } y > e^{-x}) \\
(4) & \mu \ge 0 \quad (\mu = 0 \text{ if } y < \frac{2}{3}) \\
(5) & y \ge e^{-x} \\
(6) & y \le \frac{2}{3}
\end{array}}$$

- Guess $y = e^{-x}$ and $y = \frac{2}{3}$ $\Rightarrow$ $e^{-x} = \frac{2}{3}$ $\quad -x = \ln\frac{2}{3}$ $\quad x = \ln\frac{3}{2}$

(1) gives $\lambda = 2(\ln\frac{3}{2}+\frac{1}{2})\cdot\frac{3}{2} > 0$

(2) gives $\mu = \lambda - y = 3(\ln\frac{3}{2}+\frac{1}{2}) - \frac{2}{3} > 0$ ok! $\Rightarrow$ $\boxed{(x,y) = (\ln\frac{3}{2}, \frac{2}{3}) \text{ is candidate}}$

- Guess $y = e^{-x}$ $\quad y < \frac{2}{3}$ $\Rightarrow$ $\mu = 0$

(2) gives $\lambda = e^{-x}$ (1) gives $-2(x+\frac{1}{2}) + e^{-2x} = 0$ $\quad e^{-2x} = 2x+1$ $\Rightarrow$ $x = 0$

$\Rightarrow$ $y = 1$ $\boxed{\text{Impossible}}$ Since $y < \frac{2}{3}$

- Guess $y > e^{-x}$, $y = \frac{2}{3}$ $\Rightarrow$ $\lambda = 0$ $\quad$ (2) gives $\mu = -\frac{2}{3} < 0$ $\boxed{\text{Impossible}}$

- Guess $y > e^{-x}$ $\quad y < \frac{2}{3}$ $\Rightarrow$ $\lambda = \mu = 0$ $\Rightarrow$ (2) gives $y = 0$

$\Rightarrow$ $y = 0 > e^{-x}$ Impossible since $e^{-x} > 0$ for all x. $\boxed{\text{Impossible}}$

Since only one candidate, the solution is $\underline{\underline{(x,y) = (\ln\frac{3}{2}, \frac{2}{3})}}$

b) ~~The two constraints~~ $y \ge e^{-x}$ $\Rightarrow$ $y > 0$ and $y \le \frac{2}{3}$ $\Rightarrow$ $x > 0$ (or else $e^{-x} > 1 > \frac{2}{3}$).

~~Since~~ Want x close to $-\frac{1}{2}$ and y close to 0, but cost of missing target of y half that of x
$\Rightarrow$ Most important to get x as small as possible, which occur when $y = e^{-x}$, and y max as possible, i.e. $y = \frac{2}{3}$.

$\boxed{\text{Problem 3-14}}$

NB!

a) $\min (x-2)^2+(y-2)^2$ s.t. $\begin{cases} x+y \le 2 \\ x^2-4x+y \le -2 \end{cases}$ equivalent to $\max -(x-2)^2-(y-2)^2$ given same constraints

$$\mathcal{L} = -(x-2)^2-(y-2)^2-\lambda(x+y-2)-\mu(x^2-4x+y+2)$$

(1) $\frac{\partial \mathcal{L}}{\partial x} = -2(x-2)-\lambda-2\mu x+4\mu = 0$

(2) $\frac{\partial \mathcal{L}}{\partial y} = -2(y-2)-\lambda-\mu = 0$

(3) $\lambda \ge 0$ ($\lambda = 0$ if $x+y<2$)

(4) $\mu \ge 0$ ($\mu = 0$ if $x^2-4x+y<-2$)

(5) $x+y \le 2$

(6) $x^2-4x+y \le -2$

- Non binding $\Rightarrow \lambda = \mu = 0$

$\Rightarrow$ (1) gives $x=2$ (2) gives $y=2$ contradicts (5) $\boxed{\text{Impossible}}$

- 1 binding
  - $x+y=2$ $\quad x^2-4x+y<-2$ $\Rightarrow \mu=0$

(1) gives $\lambda = -2(x-2)$ (2) gives $2(y-2)=2(x-2)$ $\Rightarrow x=y$

$\Rightarrow \boxed{x=y=1}$ But this gives (6) binding, a contradiction $\boxed{\text{Impossible}}$

  - $x+y<2$ $\quad x^2-4x+y=-2$ $\quad \lambda=0$

(2) gives $\mu=-2(y-2)$ (1) gives $-2(x-2)+4x(y-2)-8(y-2)=0$

$\Rightarrow -(x-2)+(y-2)(4-2x)=0$ $\quad$ $(x-2)=2(y-2)(x-2)$

- If $x \ne 2$ then $2(y-2)=1 \Rightarrow y=\frac{1}{2}+2$ $\quad y=\frac{5}{2} \Rightarrow x^2-4x=-\frac{9}{2}$

$x^2-4x+\frac{9}{2}=0$ No solution

- If $x=2$ then $0=0$ ok. $\Rightarrow 4-8+y=-2$ $\quad y=2$ Contradicts $x+y<2$ $\boxed{\text{Impossible}}$

- All binding $\Rightarrow x+y=2 \quad x^2-4x+y=-2$

$\Rightarrow \underline{y=2-x} \quad \Rightarrow x^2-4x+2-x=-2$

$\Rightarrow x^2-5x+4=0 \quad \Rightarrow \underline{x=4} \text{ or } \underline{x=1}$

- $x=4 \Rightarrow y=-2 \qquad x=1 \Rightarrow y=1$

Two candidates $(x_1,y_1)=(4,-2)$ and $(x_2,y_2)=(1,1)$.

~~• (4,-2) gives in (2) $\lambda=-8-\mu \leq 0$ if $\mu \geq 0$ impossible~~

~~• (1,1) gives in (2) $\lambda=-2-\mu<0$ if $\mu\geq 0$ impossible~~

- $(4,-2)$ gives in (2): $\underline{\lambda=8-\mu}$

(1) gives $-4-8+\mu-8\mu+4\mu=0 \quad 3\mu=-12<0$ $\boxed{\text{Impossible}}$

- $(1,1)$ gives in (2): $\underline{\lambda=2-\mu}$

(1) gives $2+\mu-2-2\mu+4\mu=0 \Rightarrow 3\mu=0 \quad \underline{\mu=0} \Rightarrow \underline{\lambda=2>0}$

ok! $\Rightarrow$ $\boxed{(x,y)=(1,1) \text{ is candidate}}$

Since the only candidate, $\underline{(x,y)=(1,1) \text{ is optimal solution}}$

b) It's about finding the minimal distance from the point $(x,y)=(2,2)$ given the constraints. If you draw the two constraints ($y\leq 2-x$ and $y\leq -x^2+4x-2$), then you will see that $(1,1)$ is the point ~~along~~ among all points satisfying the constraints that is closest to the point $(2,2)$.