Seminar 10

Problem by Framstad

$$J_0(x) = \max_{u_t \in [0,1]} \left\{ \sum_{t=0}^{T-1} 2q^t \sqrt{u_t x_t} + pq^T \sqrt{x_T} \right\}, \quad x_{t+1} = x_t(1-u_t) \quad x_0 = x \geq 0$$

$q \in (0,1)$, $p \in \mathbb{R}$

a) $J_T(x) = \max_{u \in [0,1]} \{ pq^T \sqrt{x} \} = \underline{pq^T \sqrt{x}}$ $\quad$ $\underline{u_T \text{ any number in } [0,1]}$

$$J_{T-1}(x) = \max_{u \in [0,1]} \{ 2q^{T-1} \sqrt{ux} + pq^T \sqrt{x(1-u)} \} = q^{T-1} \max_{u \in [0,1]} \{ 2\sqrt{ux} + pq\sqrt{x(1-u)} \}$$

$$\Rightarrow (ux)^{-\frac{1}{2}} x + \frac{1}{2} pq (x(1-u))^{-\frac{1}{2}} (-x) = 0 \qquad \text{(Indifferent if } x = 0)$$

$$x > 0 \Rightarrow \quad (1-u)^{\frac{1}{2}} = \frac{pq}{2} u^{\frac{1}{2}}$$

$$1-u = \frac{p^2 q^2}{4} u \qquad u = \frac{1}{1 + \frac{p^2 q^2}{4}} \qquad \underline{u_{T-1} = \frac{4}{p^2 q^2 + 4}} \qquad 1 - u_{T-1} = \frac{p^2 q^2}{p^2 q^2 + 4}$$

$$\Rightarrow J_{T-1}(x) = \sqrt{x}\, q^{T-1} \left(2 \frac{2}{\sqrt{p^2 q^2 + 4}} + pq \frac{pq}{\sqrt{p^2 q^2 + 4}} \right)$$

$$\underline{J_{T-1}(x) = \sqrt{4 + p^2 q^2}\; q^{T-1} \sqrt{x}}$$

Assume $J_{T-k}(x) = a_{T-k}\, q^{T-k} \sqrt{x}$

$$J_{T-(k+1)}(x) = q^{T-(k+1)} \max_{u \in [0,1]} \{ 2\sqrt{xu} + a_{T-k}\, q \sqrt{x(1-u)} \}$$

$$\Rightarrow u^{-\frac{1}{2}} + \frac{1}{2} a_{T-k}\, q\, (1-u)^{-\frac{1}{2}} (-1) = 0 \qquad (1-u)^{\frac{1}{2}} = \frac{1}{2} a_{T-k}\, q\, u^{\frac{1}{2}}$$

$$(1-u) = \frac{1}{4} a_{T-k}^2 q^2 u \qquad u_{T-(k+1)} = \frac{1}{1 + \frac{1}{4} a_{T-k}^2 q^2} = \frac{4}{4 + a_{T-k}^2 q^2} \qquad 1 - u = \frac{a_{T-k}^2 q^2}{4 + a_{T-k}^2 q^2}$$

$$\Rightarrow J_{T-(k+1)}(x) = q^{T-(k+1)} \sqrt{x} \left(\frac{2 \cdot 2}{\sqrt{4 + a_{T-k}^2 q^2}} + a_{T-k}\, q \frac{a_{T-k}\, q}{\sqrt{4 + a_{T-k}^2 q^2}} \right) = \sqrt{4 + a_{T-k}^2 q^2}\; q^{T-(k+1)} \sqrt{x}$$

$\underline{J_{T-(k+1)}(x) = a_{T-(k+1)}\, q^{T-(k+1)} \sqrt{x}}$ $\quad \Rightarrow \underline{J_{T-k}(x) = a_{T-k}\, q^{T-k} \sqrt{x}}$ by induction

$$\Rightarrow \underline{J_t(x) = a_t\, q^t \sqrt{x}}$$

b)

~~$J(x) = \max_{u\in[0,1]} \{ 2q\sqrt{ux} +$~~

$$\boxed{J(x) = \max_{u\in[0,1]} \{ 2\sqrt{ux} + q\, J(x(1-u)) \}}$$ Bellman equation

↳ notice, q is the discounting factor of this problem

(A) $J(x) = A\sqrt{x}$

$\Rightarrow$ (B) $\max_{u\in[0,1]} \{ 2\sqrt{ux} + q\, A\sqrt{x(1-u)} \}$

F.o.c. a $u^{-\frac{1}{2}} + \frac{1}{2} q A (1-u)^{-\frac{1}{2}}(-1) = 0$

$\Rightarrow u = 4q^{-2}A^{-2}(1-u)$ $\quad \frac{1}{4}q^2A^2 u = 1-u$ $\quad u = \frac{1}{1+\frac{1}{4}q^2A^2}$ $\quad \underline{u = \frac{4}{4+q^2A^2}}$

$\Rightarrow$ (C) $\sqrt{x}\left[2\frac{4}{\sqrt{4+q^2A^2}} + qA\frac{qA}{\sqrt{4+q^2A^2}} \right] = \underline{\sqrt{4+q^2A^2}\sqrt{x}}$

$\Rightarrow A\sqrt{x} = \sqrt{4+q^2A^2}\sqrt{x} \quad \Rightarrow \quad A^2 = 4 + q^2A^2 \quad A^2[1-q^2] = 4$

$A = \pm\frac{2}{\sqrt{1-q^2}} \quad A > 0 \Rightarrow \boxed{A = \frac{2}{\sqrt{1-q^2}}}$

$\Rightarrow$ $\underline{\underline{J(x) = \frac{2}{\sqrt{1-q^2}}\sqrt{x} \text{ satisfies the Bellman equation}}}$

$\boxed{8\text{-}01}$

$\max \int_0^1 (2x e^{-t} - 2x\dot{x} - \dot{x}^2)\,dt \quad x(0)=0 \quad x(1)=1$

a) $\frac{\partial F}{\partial x} = 2e^{-t} - 2\dot{x} \qquad \frac{\partial F}{\partial \dot{x}} = -2x - 2\dot{x}$

$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\dot{x} - 2\ddot{x}$

$\Rightarrow$ Euler equation $\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)$ gives $2e^{-t} - 2\dot{x} + 2\dot{x} + 2\ddot{x} = 0$

$\Rightarrow \boxed{\ddot{x} = -e^{-t}}$

b) $\dot{x} = e^{-t} + C \qquad \underline{x = -e^{-t} + Ct + D}$

$x(0) = -1 + D = 0 \quad \Rightarrow \quad \underline{D = 1}$

$x(1) = -e^{-1} + C + 1 = 1 \quad \Rightarrow \quad \underline{C = e^{-1}} \quad \Rightarrow \quad \underline{\underline{x(t) = -e^{-t} + e^{-1}t + 1}}$

$\boxed{8\text{-}02}$

$$\max \int_0^1 (4x e^{-t} - 5x^2 - \dot{x}^2) e^{-4t}\, dt \qquad x(0) = \frac{5}{3} \quad x(1) = 2e^{-1}$$

$$\frac{\partial F}{\partial x} = (4e^{-t} - 10x) e^{-4t} \qquad \frac{\partial F}{\partial \dot{x}} = -2\dot{x} e^{-4t}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\ddot{x} e^{-4t} + 8\dot{x} e^{-4t} = 2e^{-4t}(4\dot{x} - \ddot{x})$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = e^{-4t}\left[4e^{-t} - 10x - 2(4\dot{x} - \ddot{x})\right] = 0$$

$$\boxed{\ddot{x} - 4\dot{x} - 5x = -2e^{-t}} \quad \text{Euler equation}$$

$$\frac{1}{4}(-4)^2 - 5 = 4 + 5 = 9 > 0$$

$\Rightarrow$ Homogeneous solution: $x(t) = Ae^{r_1 t} + Be^{r_2 t}$ $\quad r_{1,2} = 2 \pm \sqrt{9}$

$$\underline{x(t) = Ae^{5t} + Be^{-t}}$$

Nonhomogeneous: $f(t) = -2e^{-t}$, i.e. of the form pe^{qt} where $p = -2$, $q = -1$.

Notice $q = -1$ is single root of $q^2 - 4q - 5$

$\Rightarrow$ $u^* = Cte^{-t}$. Must find C, insert in Euler equation.

$$\dot{u}^* = Ce^{-t}[1-t] \qquad \ddot{u}^* = Ce^{-t}[t-2]$$

Thus $\quad Ce^{-t}[t-2] - 4Ce^{-t}[1-t] - 5Cte^{-t} = -2e^{-t}$

$$\Rightarrow Ce^{-t}[t - 2 - 4 + 4t - 5t] = Ce^{-t}[-6] = -2e^{-t} \quad \Rightarrow \boxed{C = \frac{1}{3}}$$

Thus $\quad x(t) = Ae^{5t} + Be^{-t} + \frac{1}{3}te^{-t}$

$$x(0) = A + B = \frac{5}{3} \qquad B = \frac{5}{3} - A \qquad x(1) = Ae^{5} + \left(\frac{5}{3} - A\right)e^{-1} + \frac{1}{3}e^{-1} = 2e^{-1}$$

$$\Rightarrow A(e^{5} - e^{-1}) = 0 \quad \underline{A = 0} \quad \underline{B = \frac{5}{3}}$$

$$\underline{\underline{x(t) = \frac{1}{3}e^{-t}[5 + t]}}$$

8-03

$\max \int_0^T \left(\frac{1}{100}tx - \dot{x}^2\right)e^{-\frac{t}{10}}\,dt \quad , x(0)=0 \quad , x(T)=S$

a) $\frac{\partial F}{\partial x} = \frac{1}{100}te^{-\frac{t}{10}} \qquad \frac{\partial F}{\partial \dot{x}} = -2\dot{x}e^{-\frac{t}{10}} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\left[\ddot{x}e^{-\frac{t}{10}} + \dot{x}e^{-\frac{t}{10}}\left(-\frac{1}{10}\right)\right]$

$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \boxed{e^{-\frac{t}{10}}\left[\frac{t}{100} + 2\left(\ddot{x} - \frac{\dot{x}}{10}\right)\right] = 0}$ Euler equation

$\Rightarrow \ddot{x} - \frac{1}{10}\dot{x} = -\frac{t}{200}$ Taking integral gives $\dot{x} - \frac{1}{10}x = -\frac{t^2}{400} + C$

$\frac{d}{dt}\left(xe^{-\frac{1}{10}t}\right) = -\frac{t^2}{400}e^{-\frac{1}{10}t} + Ce^{-\frac{1}{10}t} \quad \Rightarrow xe^{-\frac{1}{10}t} = -\frac{1}{400}\int t^2e^{-\frac{1}{10}t}dt + C\int e^{-\frac{1}{10}t}dt + D$ (*)

$\int t^2e^{-\frac{1}{10}t}dt = -10t^2e^{-\frac{1}{10}t} + \int 20te^{-\frac{1}{10}t}dt = -10t^2e^{-\frac{1}{10}t} + 20\left[-10te^{-\frac{1}{10}t} + \int 10e^{-\frac{1}{10}t}dt\right]$

$= -10e^{-\frac{1}{10}t}\left[t^2 + 20t + 200\right]$

$\int e^{-\frac{1}{10}t}dt = -10e^{-\frac{1}{10}t}$

$\Rightarrow$ (*) gives $x(t) = \frac{1}{40}\left[t^2 + 20t + 200\right] - 10C + De^{\frac{1}{10}t}$

$\Rightarrow x(t) = \frac{1}{40}t^2 + \frac{1}{2}t + A + De^{\frac{1}{10}t} \qquad (A = 5 - 10C)$

b) $T=10,\ S=20 \qquad x(0) = A + D = 0 \quad A = -D$

$x(10) = \frac{5}{2} + 5 - D + De = 20 \qquad D = \frac{15 - \frac{5}{2}}{e-1} \quad \boxed{D = \frac{25}{2(e-1)}} \quad \boxed{A = -\frac{25}{2(e-1)}}$

$x(t) = \frac{1}{40}t^2 + \frac{1}{2}t - \frac{25}{2(e-1)} + \frac{25}{2(e-1)}e^{\frac{1}{10}t}$

$\frac{1}{100}tx - \dot{x}^2$ is concave w.r.t. $(x, \dot{x})$ since $\frac{\partial^2 F}{\partial x^2} = 0$, $\frac{\partial^2 F}{\partial \dot{x}^2} = -2 < 0 \quad \frac{\partial^2 F}{\partial x \partial \dot{x}} = 0$

$\Rightarrow \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial \dot{x}^2} - \left(\frac{\partial^2 F}{\partial x \partial \dot{x}}\right)^2 = 0$, thus solution is optimal.

8-06

$$\min \int_0^T p x^2 + q \frac{1}{b^2}(\dot{x} - a x)^2 \, dt \qquad x(0) = x_0 \quad x(T) = x_T$$

a) $\frac{\partial F}{\partial x} = 2px - 2q \frac{a}{b^2}(\dot{x} - ax) \qquad \frac{\partial F}{\partial \dot{x}} = 2q \frac{1}{b^2}(\dot{x} - ax)$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2q\frac{1}{b^2}(\ddot{x} - a\dot{x})$$

Eule equation: $2px - 2\frac{q a}{b^2}(\dot{x} - ax) - 2q\frac{1}{b^2}(\ddot{x} - a\dot{x}) = 0$

$$\frac{q}{b^2}\ddot{x} - px - \frac{q a^2}{b} x = 0$$

$$\boxed{\ddot{x} - x\left(\frac{p}{q}b^2 + \frac{a^2}{b}\right) = 0}$$

$\Rightarrow x(t) = Ae^{r_1 t} + Be^{r_2 t} \qquad r_{1,2} = \pm\sqrt{\frac{p}{q}b^2 + \frac{a^2}{b}}$ if $\frac{p}{q}b^2 + \frac{a^2}{b} > 0$, $x(t) = (A + Bt)$ if $\frac{p}{q}b^2 + \frac{a^2}{b} = 0$

b) ~~[illegible]~~ $p = 0 \quad q = a = b = T = 1 \quad x_0 = 0 \quad x_T = 1 \quad \Rightarrow \frac{p}{q}b^2 + \frac{a^2}{b} > 0$

$r_{1,2} = \pm\sqrt{1} = \pm 1 \quad \Rightarrow x(t) = Ae^{t} + Be^{-t}$

$x(0) = A + B = 0 \qquad A = -B$

$x(1) = B(-e + e^{-1}) = 1 \qquad B = \frac{1}{e^{-1} - e} \qquad A = \frac{1}{e - e^{-1}}$

$x(t) = \frac{1}{e - e^{-1}}\left[e^{t} - e^{-t}\right] \qquad x(t) = \frac{e^t - e^{-t}}{e - e^{-1}}$

Problem is $\min \int_0^1 (\dot{x} - x)^2 \, dt \quad \Rightarrow \frac{\partial^2 F}{\partial x^2} = 2 \quad \frac{\partial^2 F}{\partial \dot{x}^2} = 2 \quad \frac{\partial^2 F}{\partial x \partial \dot{x}} = -2 \quad \Rightarrow$ Problem convex

Ok since mi

$\boxed{9\text{-}01}$

$\max\limits_{u\in\mathbb{R}} \int_0^2 (2x-3u-\alpha u^2)\,dt \qquad \dot{x} = x+u \qquad x(0)=5 \qquad x(2)$ free

$\alpha>0$

$H(t,x,u,p) = 2x-3u-\alpha u^2 + p(x+u)$

$H'_x = 2+p \Rightarrow \underline{\dot{p} = -2-p}$

Thus $\dot{p}+p=-2 \qquad \frac{d}{dp}(pe^{t}) = -2e^{t} \qquad pe^{t} = -2e^{t}+C \qquad \underline{p = Ce^{-t}-2}$

$p(2) = Ce^{-2}-2 = 0 \qquad \underline{C = 2e^{2}} \Rightarrow \boxed{p(t) = 2(e^{2-t}-1)} \qquad \underline{p \geq 0\ \forall t}$

$H'_u = -3-2\alpha u+p = 0 \Rightarrow u = \frac{p-3}{2\alpha} \qquad u(t) = \frac{e^{2-t}-1-\frac{3}{2}}{\alpha} \qquad \boxed{u(t) = \frac{e^{2-t}-\frac{5}{2}}{\alpha}}$

$\dot{x} = x + \frac{e^{2-t}-\frac{5}{2}}{\alpha} \qquad \dot{x}-x = \frac{e^{2-t}-\frac{5}{2}}{\alpha}$

$\frac{d}{dt}(xe^{-t}) = \frac{e^{2-2t}-\frac{5}{2}e^{-t}}{\alpha} \qquad xe^{-t} = \frac{\frac{e^{2-2t}}{-2}+\frac{5}{2}e^{-t}}{\alpha}+C \qquad \underline{x = \frac{5}{2\alpha}+Ce^{t}-\frac{1}{2\alpha}e^{2-t}}$

$x(0) = \frac{5}{2\alpha}+C-\frac{1}{2\alpha}e^{2} = 5 \qquad \underline{C = 5+\frac{e^{2}}{2\alpha}-\frac{5}{2\alpha} = \frac{e^{2}+10\alpha-5}{2\alpha}}$

$$\boxed{x(t) = \frac{5}{2\alpha}+\frac{e^{2}+10\alpha-5}{2\alpha}e^{t}-\frac{1}{2\alpha}e^{2-t}}$$

Thus, $\underline{(x(t),u(t)) = \left(\frac{5}{2\alpha}+\frac{e^{2}+10\alpha-5}{2\alpha}e^{t}-\frac{1}{2\alpha}e^{2-t},\ \frac{e^{2-t}-\frac{5}{2}}{\alpha}\right) \text{ and } p(t) = 2(e^{2-t}-1)}$