Seminar 11 - ECON 4140

Problem 9-02

$$\max_{u \in \mathbb{R}} \int_0^T -(x-u+2)^2 e^{-rt}\,dt \ , \quad \dot{x} = u(t) - \delta x(t) \ , \quad x(0) = x_0 \ , \quad x(T) = x_T$$

$T, r, \delta, x_0, x_T > 0.$

a) $H(t,x,u,p) = -(x-u+2)^2 e^{-rt} + p(u - \delta x)$

$$\boxed{\begin{array}{l} u \text{ maximizes } H(t,x^*,u,p) \\ \dot{p} = -H^*_x \\ p(T) \text{ free} \end{array}}$$

b)

$H'_x = -2(x-u+2)e^{-rt} - p\delta \qquad H''_{xx} = -2e^{-rt} < 0$

$H'_u = 2(x-u+2)e^{-rt} + p \qquad H''_{uu} = -2e^{-rt} < 0$

$H''_{xu} = 0 \qquad \Rightarrow H''_{xx}H''_{uu} - (H''_{xu})^2 = 4e^{-2rt} > 0 \qquad \Rightarrow$ Concave in (x,u)

c) $H(t,x,u,p) = -(x-u+2)^2 e^{-0,1t} + p(u - 0,5x)$

$$\boxed{\begin{array}{lll} u & \left\{ u \text{ maximizes } H(t,x^*,u,p) \right. & \#1 \\ p & \left\{ \begin{array}{l} \dot{p} = 2(x-u+2)e^{-0,1t} + 0,5p \\ p(T) \text{ free} \end{array} \right. & \#2 \\ x & \left\{ \begin{array}{l} \dot{x} = u - 0,5x \\ x(0) = 0 \\ x(10) = 8 \end{array} \right. & \begin{array}{l} \#3 \\ \#4 \\ \#5 \end{array} \end{array}}$$

H concave in u $\Rightarrow$ (#1) can use first order condition to find u

$2(x-u+2)e^{-0,1t} + p = 0$

$\Rightarrow u = x + 2 + \frac{1}{2}e^{0,1t}p$

(#2) $\Rightarrow \dot{p} = 2\left(-\frac{1}{2}e^{0,1t}p\right)e^{-0,1t} + 0,5p = -0,5p$

$\dot{p} + 0,5p = 0 \qquad \frac{d}{dt}\left(pe^{0,5t}\right) = 0$

$p = Ce^{-0,5t}$

(#3): $\dot{x} = x + 2 + \frac{1}{2}e^{0,1t}\,Ce^{-0,5t} - 0,5x = 0,5x + 2 + \frac{1}{2}Ce^{-0,4t}$

$\dot{x} - 0,5x = 2 + \frac{1}{2}Ce^{-0,4t}$ $\quad \frac{d}{dt}(xe^{-0,5t}) = 2e^{-0,5t} + \frac{1}{2}Ce^{-0,9t}$

$xe^{-0,5t} = -4e^{-0,5t} - \frac{5}{9}Ce^{-0,9t} + D$

$\underline{x(t) = -4 - \frac{5}{9}Ce^{-0,4t} + De^{0,5t}}$

(#4) $x(0) = -4 - \frac{5}{9}C + D = 0$ $\quad \underline{D = 4 + \frac{5}{9}C}$

(#5) $x(10) = -4 - \frac{5}{9}Ce^{-4} + (4 + \frac{5}{9}C)e^{5} = 8$

$\frac{5}{9}C[e^{5} - e^{-4}] = 12 - 4e^{5}$ $\quad \underline{\frac{5}{9}C = \frac{12-4e^{5}}{e^{5}-e^{-4}}}$ $\quad D = 4 + \frac{12-4e^{5}}{e^{5}-e^{-4}} = \underline{\frac{12-4e^{-4}}{e^{5}-e^{-4}}}$

$$\boxed{x^*(t) = -4 + \frac{4e^{5}-12}{e^{5}-e^{-4}}e^{-0,4t} + \frac{12-4e^{-4}}{e^{5}-e^{-4}}e^{0,5t}}$$

$$\boxed{p(t) = \frac{9}{5}\,\frac{12-4e^{5}}{e^{5}-e^{-4}}\,e^{-0,5t}} \qquad \boxed{u(t) = x(t) + 2 + \frac{1}{2}e^{0,1t}\,p(t)}$$

Problem 9-03

a) $\max_{u\in\mathbb{R}} \int_0^2 (x(t)-u(t)^2)\,dt$, $\dot{x}(t)=x(t)+u(t)$, $x(0)=0$, $x(2)$ free

$H(t,x,u,p) = x-u^2+p(x+u)$ $\qquad H'_x = 1+p$

$$\boxed{\begin{array}{lll} u & \{ u\in\mathbb{R} \text{ maximizes } H(t,x^*,u,p) & (1) \\ p & \begin{cases} \dot{p} = -H_x & (2) \\ p(2)=0 & (3) \end{cases} & \\ x & \begin{cases} \dot{x} = x+u & (4) \\ x(0)=0 & (5) \end{cases} & \end{array}}$$

H concave in u ⇒ can use F.O.C.

(1) $\frac{dH}{du} = -2u+p = 0 \Rightarrow \underline{u=\frac{1}{2}p}$

(2) $\dot{p} = -1-p \Rightarrow \dot{p}+p=-1 \quad \frac{d}{dt}(pe^t) = -e^t \quad pe^t = -e^t + C$

$\underline{p(t) = Ce^{-t}-1}$

(3): $p(2) = Ce^{-2}-1=0 \quad \underline{C=e^2} \Rightarrow \boxed{p(t) = e^{2-t}-1}$

$\Rightarrow \boxed{u^*(t) = \frac{1}{2}\left(e^{2-t}-1\right)}$

(4): $\dot{x} = x+\frac{1}{2}(e^{2-t}-1) \quad \dot{x}-x = \frac{1}{2}(e^{2-t}-1) \quad \frac{d}{dt}(xe^{-t}) = \frac{1}{2}(e^{2-2t}-e^{-t})$

$xe^{-t} = -\frac{1}{4}e^{2-2t}+\frac{1}{2}e^{-t}+D \quad \underline{x(t) = -\frac{1}{4}e^{2-t}+De^t+\frac{1}{2}}$

(5): $x(0) = -\frac{1}{4}e^2+D+\frac{1}{2}=0 \quad \underline{D=\frac{1}{4}e^2-\frac{1}{2}}$

$\Rightarrow \boxed{x^*(t) = -\frac{1}{4}e^{2-t}+\left(\frac{1}{4}e^2-\frac{1}{2}\right)e^t+\frac{1}{2}}$

$x-u^2$ concave in (x,u) ⇒ know candidate is optimal solution.

b) Here, (1) might change.

From a), $u(t) = \frac{1}{2}(e^{2-t}-1)$

we need to concider if this might not be in $[0,1]$ for $t \in [0,2]$

Checks $u(0) = \frac{1}{2}(e^2-1) \approx 3{,}1945 > 1$ problem!

Ask: at what time t^* does $u(t)$ start to be in $[0,1]$?

$u(t^*) = \frac{1}{2}(e^{2-t^*}-1) = 1$ $\quad e^{2-t^*} = 3 \quad 2-t^* = \ln 3 \quad \underline{t^* = 2-\ln 3} \approx 0{,}9$

i.e. for $t \in [2-\ln 3, 2]$ F.O.C. will give $u(t) \in [0,1]$ i.e. $u(t) = \frac{1}{2}(e^{2-t}-1)$.

For $t \in [0, 2-\ln 3)$ we want $u(t)$ as close to this as possible, i.e. choose corner solution $u(t) = 1$.

Thus, $u(t) = \begin{cases} \frac{1}{2}(e^{2-t}-1) & t \in [2-\ln 3, 2] \\ 1 & t \in [0, 2-\ln 3) \end{cases}$

In the case $u=1$ we get of (4): $\dot{x} = x+1 \Rightarrow \frac{d}{dt}(xe^{-t}) = e^{-t}$

$\Rightarrow xe^{-t} = -e^{-t} + C \quad x = Ce^t - 1 \quad x(0) = C-1 = 0 \Rightarrow \underline{C=1}$

$\Rightarrow \underline{x(t) = e^t - 1}$

~~Thus $x(t) = \begin{cases} \text{as in part a) for } t \in [2-\ln 3, 2] \\ e^t - 1 \text{ for } t \in [0, 2-\ln 3) \end{cases}$~~

After time $t = 2-\ln 3$, $x(t) = -\frac{1}{4}e^{2-t} + De^t + \frac{1}{2}$, but constant D is determined now by the continuity of $x(t)$. This means the two x's derived must be equal at time $2-\ln 3$. Thus: $e^{2-\ln 3} - 1 = \underbrace{-\frac{1}{4}e^{\ln 3}}_{=-\frac{3}{4}} + De^{2-\ln 3} + \frac{1}{2}$

$e^{2-\ln 3} - \frac{3}{4} = De^{2-\ln 3}$

$D = 1 - \frac{3}{4}e^{\ln 3 - 2} = \underline{1 - \frac{9}{4}e^{-2}}$ $\quad \Rightarrow \boxed{x(t) = \begin{cases} e^t - 1 & \text{for } t \in [0, 2-\ln 3] \\ -\frac{1}{4}e^{2-t} + (1-\frac{9}{4}e^{-2})e^t + \frac{1}{2} & \text{for } t \in (2-\ln 3, 2] \end{cases}}$

Problem 9-05

$$\max \int_0^T (ax^2 + 2bx\dot{x} + c\dot{x}^2 + dt^2\dot{x})e^{-rt}\,dt \quad , x(0) = x_0 \ , x(T) = x_T$$

a) $f(x,y) = ax^2 + 2bxy + cy^2 + dt^2 y$

$\frac{\partial f}{\partial x} = 2ax + 2by \quad \frac{\partial f}{\partial y} = 2bx + 2cy + dt^2$

$\frac{\partial^2 f}{\partial x^2} = 2a \quad \frac{\partial^2 f}{\partial x \partial y} = 2b \quad \frac{\partial^2 f}{\partial y^2} = 2c$

Need $a \le 0$, $b \le 0$, $4ac - 4b^2 \ge 0 \qquad ac \ge b^2$

Thus concave if $\boxed{a \le 0, \cancel{b} \le 0, ac \ge b^2}$ $\boxed{r \text{ doesn't matter}}$

b) $\frac{\partial F}{\partial x} = (2ax + 2b\dot{x})e^{-rt} \qquad \frac{\partial F}{\partial \dot{x}} = (2bx + 2c\dot{x} + dt^2)e^{-rt}$

$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = (2b\dot{x} + 2c\ddot{x} + 2dt)e^{-rt} - r(2bx + 2c\dot{x} + dt^2)e^{-rt}$

Thus, $\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2e^{-rt}\left[ax + b\dot{x} + r(bx + c\dot{x} + \frac{1}{2}dt^2) - b\dot{x} - c\ddot{x} - dt\right] = 0$

$\boxed{\ddot{x} - r\dot{x} - \frac{(a+rb)}{c}x = \frac{1}{2}\frac{r}{c}dt^2 - \frac{d}{c}t}$ Euler equation

c) $a = -9$, $b = 1$, $c = -1$, $d = 3$, $x_0 = 0$, $x_T = 0$, $T = 1$, $r = 0$ $\Rightarrow$ Problem concave

Euler eq. gives $\boxed{\ddot{x} - 9x = 3t}$

$\Rightarrow x(t) = Ae^{r_1 t} + Be^{r_2 t} + u^*$ where $r_{1,2} = \pm 3$

$u^* = Ct + D \quad \Rightarrow \dot{u}^* = C \quad \ddot{u}^* = 0$

$\Rightarrow -9(Ct + D) = 3t \quad \Rightarrow \underline{C = -\frac{1}{3}} \quad \underline{D = 0} \qquad x(t) = Ae^{3t} + Be^{-3t} - \frac{1}{3}t$

$x(0) = A + B = 0 \quad \underline{B = -A} \qquad x(1) = A\left[e^{3} - e^{-3}\right] - \frac{1}{3} = 0 \qquad A = \frac{1}{3(e^{3} - e^{-3})}$

$$x(t) = \frac{e^{3t}}{3(e^3-e^{-3})} - \frac{e^{-3t}}{3(e^3-e^{-3})} - \frac{1}{3}t$$

d) $\max_{u \in \mathbb{R}} \int_0^1 (-9x^2 + 2xu - u^2 + 3t^2u)\,dt$, $\dot{x} = u$, $x(0)=0$, (i) $x(1)$ free (ii) $x(1) \geq 2$

$H(t,x,u,p) = -9x^2 + 2xu - u^2 + 3t^2u + pu$

(1) u maximizes $H(t,x^*,u,p)$

(2) $\dot{p} = -H_x = 18x - 2u$

(3) $p(1) \begin{cases} = 0 & \text{(i)} \\ \geq 0 \ (p(1)=0 \text{ if } x(1)>2) & \text{(ii)} \end{cases}$

(4) $\dot{x} = u$

(5) $x(0) = 0$

(6) $x(1)$ free (i)
$x(1) \geq 2$ (ii)

OBS! This part is not necessary, one knows that the Euler equation must still hold in the transformed problem. I show this formally here just for convenience.
You might skip right to (*)

(1) gives F.o.c. $2x - 2u + 3t^2 + p = 0$ $\quad u = x + \frac{3}{2}t^2 + \frac{1}{2}p$

$H'_x = -18x + 2u$

(2) gives $\dot{p} = 18x - 2u = 18x - 2x - 3t^2 - p$ $\quad \dot{p} + p = 16x - 3t^2$

By (4) we thus have $\begin{cases} \dot{x} = x + \frac{3}{2}t^2 + \frac{1}{2}p \\ \dot{p} = -p - 3t^2 + 16x \end{cases}$ system of differential equations

We get $p = 2\dot{x} - 2x - 3t^2$ $\quad \dot{p} = 2\ddot{x} - 2\dot{x} - 6t$

Incert in below equation: $2\ddot{x} - 2\dot{x} - 6t = -2\dot{x} + 2x + 3t^2 - 3t^2 + 16x$

$\boxed{\ddot{x} - 9x = 3t}$ The Euler equation! (*)

$\Rightarrow x(t) = Ae^{3t} + Be^{-3t} - \frac{1}{3}t$ $\quad x(0) = A + B = 0$ $\quad B = -A$

$x(t) = A(e^{3t} - e^{-3t}) - \frac{1}{3}t$

$p(t) = -\frac{\partial F}{\partial \dot{x}} = -\frac{\partial F}{\partial u} = -2x + 2\dot{x} - 3t^2$

$$p = -2x + 2\left(A\left(3e^{3t} + 3e^{-3t}\right) - \tfrac{1}{3}\right) - 3t^2 = \underline{-2x + 6A\left(e^{3t} + e^{-3t}\right) + \tfrac{2}{3} - 3t^2}$$

(i) $x(1)$ free $\Rightarrow$ $p(1) = 0$

$$p(1) = -2x(1) + 6A\left(e^{3} + e^{-3}\right) + \tfrac{2}{3} - 3$$

$$= -2A\left(e^{3} - e^{-3}\right) + \tfrac{2}{3} + 6A\left(e^{3} + e^{-3}\right) + \tfrac{2}{3} - 3 = 0$$

$$A\left[4e^{3} + 8e^{-3}\right] = +\tfrac{5}{3} \qquad \underline{A = +\frac{5}{12\left(e^{3} + 2e^{-3}\right)}} \qquad \underline{B = \frac{-5}{12\left(e^{3} + 2e^{-3}\right)}}$$

$$\Rightarrow \underline{x(t) = +\frac{5}{12\left(e^{3} + 2e^{-3}\right)}\left(e^{3t} - e^{-3t}\right) - \tfrac{1}{3}t} \qquad (\mathrm{I})$$

(ii) $x(1) \geq 2$

- case 1 $x(1) = 2$ $\Rightarrow$ $x(1) = A\left(e^{3} - e^{-3}\right) - \tfrac{1}{3} = 2$ $\quad A = \dfrac{7}{3\left(e^{3} - e^{-3}\right)}$

$$\Rightarrow \underline{x(t) = \frac{7}{3\left(e^{3} - e^{-3}\right)}\left(e^{3t} - e^{-3t}\right) - \tfrac{1}{3}t} \qquad (\mathrm{II})$$

- case 2 $x(1) > 2$ $\Rightarrow$ $p(1) = 0$ $\Rightarrow$ gets (I) $\Rightarrow$ $x(1) < 1$, a contradiction

  Thus one chooses such that $x(1) = 2$ and arrives at (II)

To summarize: $\boxed{\begin{array}{l}\text{with (i) one gets (I)} \\ \text{with (ii) one gets (II)}\end{array}}$

Then $u(t)$ follows by $u = \dot{x}$

Problem 9-07

$$\max_{u\in[0,1]} \int_0^T (1-u)x^2\,dt \qquad \dot{x} = ux \quad , x(0)=1 \quad x(T) \text{ free}$$

a) $H(t,x,u,p) = (1-u)x^2 + pux \qquad H'_x = 2(1-u)x + pu$

$$\boxed{\begin{array}{l} \#1 \quad u^* \text{ maximizes } (1-u)x^2 + pux \\ \#2 \begin{cases} \dot{p} = -2(1-u)x - pu \\ p(T) = 0 \end{cases} \\ \#3 \begin{cases} \dot{x} = ux \\ x(0) = 1 \end{cases} \end{array}}$$

b) $\dot{x} = ux$ since $u\in[0,1]$ and $x(0)=1$ we get that $\dot{x} \geq 0$

$\Rightarrow x(t) \geq 1$

$\dot{p} = \underbrace{-2(\overset{\geq 0}{1-u})\overset{\geq 1}{x}}_{\leq 0} - p\overset{\geq 0}{u}$ $\quad \dot{p}(T) = -2(1-u(T))x(T) \leq 0$. Thus $p(t) \geq 0$ must be true close to time T but then we see that $\dot{p} \leq 0$ is true here as well, and will thus always need to be true $\Rightarrow p(t) \geq 0$.

~~$\Rightarrow p(t)$ must be ≥ 0 close to T $\Rightarrow \dot{p} = -2(1-u)x - pu \leq 0$ close to T~~

Assume $p(t) = 0$ for any $t < T$. Then $\dot{p} = -2(1-u)\overset{\geq 1}{x}$ must equal zero, or we will forever get negative p. That means $u = 1$, but that contradicts #1.

Thus $p(t) > 0$ for all $t < T$. Thus, since $x \geq 1$ ~~[illegible]~~ and $u\in[0,1]$ we see that $\dot{p} < 0$ must be

c) We know $p(t) \geq 0$ for all times, and is strictly decreasing

- If $p(0) \leq 1$ $p(t) \leq 1$ for all times $\Rightarrow u = 0$ always since $x(t) \geq 1$.

Then we get $\dot{p} = -2x$ and $\dot{x} = 0$ $\Rightarrow \dot{p} = -2$ $p(t) = -2t + C$

$p(T) = 0 \Rightarrow -2T + C = 0$ $\quad C = 2T$ $\quad p(t) = 2(T-t)$

$\Rightarrow p(0) = 2T > 2 \cdot \frac{1}{2} = 1$ A contradiction!

- Thus $p(0) > 1$ must hold. Then we know we start by $u = 1$ because of #1. Since $p(T) = 0$ and $x(t) \geq 1$ there is a time t^* when $p(t^*) = x(t^*)$. After that time $p(t) < x(t)$ since $\dot{p} < 0$, $\dot{x} \geq 0$
$\Rightarrow$ change strategy to $u = 0$ (because of #1). Must find this t^*.

$$\boxed{u(t) = \begin{cases} 1 & t \in [0, t^*] \\ 0 & t \in (t^*, T] \end{cases}} \Rightarrow \dot{x} = \begin{cases} x & t \in [0, t^*] \\ 0 & t \in (t^*, T] \end{cases}$$

$x(0) = 1 \Rightarrow x(t) = e^t$

$$\Rightarrow \boxed{x(t) = \begin{cases} e^t & t \in [0, t^*] \\ D & t \in (t^*, T] \end{cases}}$$

Must be equal at time t^* (since $x(t)$ continuous)

$\Rightarrow x(t^*) = \boxed{e^{t^*} = D}$

$$\dot{p} = \begin{cases} -p & t \in [0, t^*] \\ -2D & t \in (t^*, T] \end{cases} \quad \begin{aligned} &\Rightarrow p(t) = Ee^{-t} \\ &\Rightarrow p(t) = -2Dt + F = -2e^{t^*}t + F \end{aligned}$$

$p(T) = -2e^{t^*}T + F = 0$ $\quad \underline{F = 2e^{t^*}T}$ $\quad \Rightarrow \underline{p(t) = 2e^{t^*}(T-t) \text{ for } t \in (t^*, T]}$

$p(t^*) = Ee^{-t^*} = 2e^{t^*}(T-t^*)$ $\Rightarrow \underline{E = 2e^{2t^*}(T-t^*)}$ $\Rightarrow \underline{p(t) = 2e^{2t^*}(T-t^*)e^{-t} \text{ for } t \in [0, t^*]}$

$x(t^*) = p(t^*)$ by definition of t^* $\Rightarrow 2e^{t^*}(T-t^*) = e^{t^*}$ $\Rightarrow T - t^* = \frac{1}{2}$ $\quad \underline{\underline{t^* = T - \frac{1}{2}}}$

Thus, (x, u) and p is as above with $t^* = T - \frac{1}{2}$.

Problem 9-10

$$\max_{u\in[0,\infty)} \int_0^1 -(x-u-a)^2\,dt \qquad \dot{x}=u-x \qquad x(0)=1 \qquad x(1) \text{ free}$$

a) $H(t,x,u,p) = -(x-u-a)^2 + p(u-x)$ $\qquad H'_x = -2(x-u-a)-p$

$$\boxed{\begin{array}{ll} (1) & \{ u^* \text{ maximizes } H(t,x,u,p) \\ (2) & \begin{cases} \dot{p} = 2(x-u-a)+p \\ p(1)=0 \end{cases} \\ (3) & \\ (4) & \begin{cases} \dot{x} = u-x \\ x(0)=1 \end{cases} \\ (5) & \end{array}}$$

(1) gives F.o.c. $2(x-u-a)+p=0$ $\quad \underline{u = x-a+\frac{1}{2}p}$

(2) $\dot{p} = 2(x-x+a-\frac{1}{2}p-a)+p = 0 \Rightarrow \underline{p(t)=C}$

(3) $p(1)=0 \Rightarrow \boxed{p(t)=0} \Rightarrow \underline{u=x-a}$

(4) $\dot{x} = x-a-x = -a \Rightarrow \underline{x(t) = -at+D}$

(5) $x(0) = \underline{D=1} \Rightarrow \boxed{x(t)=1-at}$

$\Rightarrow \boxed{u(t) = 1-a(t+1)}$

$a=4 \Rightarrow u(t) = 1-4t-4 = -4t-3 < 0 \Rightarrow$ Choose corner solution $\underline{u=0}$ (Remember $u\in[0,\infty)$!).

H is concave for all a because $-z^2$ is concave and decreasing and $x-u-a$ is convex ~~and p(u-x) is concave~~ $\Rightarrow -(x-u-a)^2$ concave by result of concavity; $p(u-x)$ linear and thus concave $\Rightarrow$ H concave. Mangasarian ok!

b) $a = \frac{1}{3} \Rightarrow u(t) = 1 - \frac{1}{3}t - \frac{1}{3} = \frac{2}{3} - \frac{1}{3}t$

$u(t) \geq 0$ when $\frac{2}{3} \geq \frac{1}{3}t$ i.e. when $2 \geq t$ Always holds since $t \in [0,1]$

~~$\Rightarrow u(t) = 1 - \frac{1}{3}(t+1)$~~ $\Rightarrow \underline{u(t) = \frac{2}{3} - \frac{1}{3}t}$

Problem 9-17

a) $\max_{u \in \mathbb{R}} \int_0^1 x + u \, dt \quad \dot{x} = 1 - \frac{1}{2}u^2 \quad x(0) = 0 \quad x(1) \geq 0$

$H(t, x, u, p) = x + u + p(1 - \frac{1}{2}u^2) \qquad H'_x = 1$

(1) u^* maximizes $x + u + p(1 - \frac{1}{2}u^2)$

(2) $\dot{p} = -1$

(3) $p(1) \geq 0$ ($p(1) = 0$ if $x(1) > 0$)

(4) $\dot{x} = 1 - \frac{1}{2}u^2$

(5) $x(0) = 0$

(6) $x(1) \geq 0$

(2) gives $\underline{p = -t + C}$

(1) gives F.O.C. $1 - pu = 0 \Rightarrow u = \frac{1}{p} \qquad \underline{u = \frac{1}{C - t}}$

(4) gives $\dot{x} = 1 - \frac{1}{2(C-t)^2} \qquad \underline{x(t) = t - \frac{1}{2(C-t)} + D}$

(5) gives $x(0) = -\frac{1}{2C} + D = 0 \qquad \underline{D = \frac{1}{2C}} \Rightarrow \underline{x(t) = t - \frac{1}{2(C-t)} + \frac{1}{2C}}$

- Assume $x(1) = 0 \Rightarrow x(1) = 1 - \frac{1}{2(C-1)} + \frac{1}{2C} = 0 \Rightarrow 2C(C-1) - 2C + 2(C-1) = 0$

$4C^2 - 4C - 2 = 0 \quad C^2 - C - \frac{1}{2} = 0 \quad C = \frac{1 \pm \sqrt{1+2}}{2} = \underline{\frac{1}{2} \pm \frac{\sqrt{3}}{2}}$

Since $p(1) \geq 0$ must have $C = \frac{1}{2} + \frac{\sqrt{3}}{2}$

thus $x(t) = t - \frac{1}{1+\sqrt{3}-2t} + \frac{1}{1+\sqrt{3}}$, $u(t) = \frac{1}{\frac{1}{2}+\frac{\sqrt{3}}{2}-t}$

$p(t) = \frac{1}{2}+\frac{\sqrt{3}}{2}-t$

• Assume $x(1)>0 \Rightarrow p(1)=0 \Rightarrow C-1=0 \Rightarrow \underline{C=1}$

$\Rightarrow \underline{p(t) = 1-t}$

$\Rightarrow u(t) = \frac{1}{1-t}$ $\quad x(t) = t - \frac{1}{2(1-t)} + \frac{1}{2}$

$\Rightarrow \lim_{t\to 1} x(t) = -\infty$ impossible!

$\Rightarrow$ The case with $x(1)=0$ occurs

$\Rightarrow$ $\underline{x(t) = t - \frac{1}{1+\sqrt{3}-2t} + \frac{1}{1+\sqrt{3}}}$ $\quad \underline{u(t) = \frac{1}{\frac{1}{2}+\frac{\sqrt{3}}{2}-t}}$ $\quad \underline{p(t) = \frac{1}{2}+\frac{\sqrt{3}}{2}-t}$

b) Here $u(0) = \frac{2}{1+\sqrt{3}} < 1$ $\quad u(1) = \frac{2}{\sqrt{3}-1} > 1$

$\Rightarrow$ There is a t^* when $u(t^*)=1$

$\frac{1}{2}+\frac{\sqrt{3}}{2}-t^* = 1$ $\quad t^* = \frac{\sqrt{3}-1}{2}$ $\quad$ For $t\in[t^*,1]$ $u(t) = \frac{1}{\frac{1}{2}+\frac{\sqrt{3}}{2}-t}$

For $t\in[0,t^*]$ choose corner solution $u=1$

$\Rightarrow \dot{x} = \frac{1}{2}$ $\quad x = \frac{1}{2}t + C$ $\quad x(0)=0 \Rightarrow x(t) = \frac{1}{2}t$

$p = -t + D$ $\quad p(t^*) = D - \frac{\sqrt{3}-1}{2} = \frac{1}{2}+\frac{\sqrt{3}}{2} - \left(\frac{\sqrt{3}-1}{2}\right)$

$\Rightarrow D = \frac{1}{2}+\frac{\sqrt{3}}{2} \Rightarrow p(t) = \frac{1}{2}+\frac{\sqrt{3}}{2}-t$ for all $t\in[0,1]$

b) Here, $u(0) = \frac{2}{2+\sqrt{3}} < 1$, $u(1) = \frac{2}{\sqrt{3}-1} > 1$

$\Rightarrow$ There is a t^* when $u(t^*) = 1$.

- In $t \in [0, t^*]$ will choose corner solution $u(t) = 1$

Thus, for $t \in [0, t^*]$ $\underline{u(t) = 1}$, $\dot{x} = \frac{1}{2} \Rightarrow x(t) = \frac{1}{2}t + E$ $x(0) = 0 \Rightarrow E = 0$

$\Rightarrow \underline{x(t) = \frac{1}{2}t}$ $\underline{p(t) = -t + C}$ as befor

- For $t \in [t^*, 1]$ $\underline{u(t) = \frac{1}{C-t}}$ as befor, $\underline{p(t) = -t + C}$

$x(t) = t - \frac{1}{2(C-t)} + D$

$x(1) = 0 \Rightarrow D = \frac{1}{2(C-1)} - 1 \Rightarrow \underline{x(t) = t - \frac{1}{2(C-t)} + \frac{1}{2(C-1)} - 1}$

$u(t^*) = 1 \Rightarrow \frac{1}{C-t^*} = 1$ $C - t^* = 1$ $\underline{C = 1 + t^*}$

$\Rightarrow x(t) = t - \frac{1}{2(1+t^*-t)} + \frac{1}{2t^*} - 1$

$x(t)$ continuous at t^* gives $t^* - \frac{1}{2} + \frac{1}{2t^*} - 1 = \frac{1}{2}t^*$

$\Rightarrow t^{*2} - 3t^* + 1 = 0 \Rightarrow t^* = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$ only the one with $-$ reason because need to be in $[0, 1]$

$\Rightarrow \boxed{t^* = \frac{3-\sqrt{5}}{2}} \Rightarrow \boxed{C = \frac{5-\sqrt{5}}{2}}$

So, $\boxed{u(t) = \begin{cases} 1 & \text{for } t \in [0, t^*] \\ \frac{1}{\frac{5-\sqrt{5}}{2} - t} & \text{for } t \in (t^*, 1] \end{cases}}$ $\boxed{x(t) = \begin{cases} \frac{1}{2}t & \text{for } t \in [0, t^*] \\ t - \frac{1}{2(\frac{5-\sqrt{5}}{2} - t)} + \frac{1}{3-\sqrt{5}} - 1 & \text{for } t \in (t^*, 1] \end{cases}}$

$\boxed{p(t) = -t + \frac{5-\sqrt{5}}{2}}$ $\boxed{t^* = \frac{3-\sqrt{5}}{2}}$