SEMINAR 6 - ECON4140

Problem 4-05

$$\int_0^1\left(\int_0^1 x e^y\,dy\right)dx = \int_0^1\left[\Big|_0^1 x e^y\right]dx = \int_0^1 x(e-1)\,dx = \Big|_0^1 \tfrac12 x^2 (e-1) = \tfrac12 (e-1)$$

Problem 4-06

$$\int_0^1\int_0^4 (\sqrt{xy}+2x+y)\,dy\,dx = \int_0^1\left[\Big|_0^4 \sqrt{x}\,\tfrac23 y^{\frac32} + 2xy + \tfrac12 y^2\right]dx = \int_0^1 \tfrac{16}{3}\sqrt{x}+8x+8\,dx = \Big|_0^1 \tfrac{32}{9}x^{\frac32}+4x^2+8x = 12+\tfrac{32}{9}$$

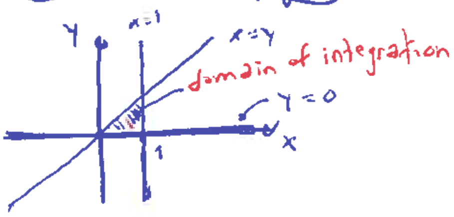
Problem 4-07

$$\int_0^1\int_0^1 x\sqrt{x^2+y}\,dx\,dy \qquad A=\int_0^1 x\sqrt{x^2+y}\,dx \qquad u=x^2+y \quad \frac{du}{dx}=2x \quad dx=\frac{du}{2x}$$

$$\Rightarrow A=\int_y^{1+y}\tfrac12\sqrt{u}\,du = \Big|_y^{1+y}\tfrac12 u^{\frac32}\tfrac23 = \tfrac13\left[(1+y)^{\frac32}-y^{\frac32}\right]$$

$$\int_0^1 A\,dy = \int_0^1 \tfrac13\left[(1+y)^{\frac32}-y^{\frac32}\right]dy = \Big|_0^1 \tfrac{2}{15}(1+y)^{\frac52}-\tfrac{2}{15}y^{\frac52} = \tfrac{2}{15}2^{\frac52}\overset{-\frac{2}{15}}{-}\tfrac{2}{15} = \tfrac{2}{15}(\sqrt{32}-2)$$

Problem on webpage



$$\Rightarrow \int_0^1\left(\int_0^x x e^y\,dy\right)dx = \int_0^1\left[\Big|_0^x xe^y\right]dx = \int_0^1 xe^x - x\,dx$$

$$= \Big|_0^1 xe^x \overset{-|_0^1\frac12x^2}{-}\int_0^1 e^x\,dx = e \overset{-\frac12}{-}\Big|_0^1 e^x = e-e+1 \overset{-\frac12}{=} \tfrac12$$

~~Opposite: $\int_0^1\left(\int_0^y xe^y\,dx\right)dy = \int_0^1\left[\Big|_0^y \tfrac12 x^2 e^y\right]dy = \int_0^1 \tfrac12 y^2 e^y\,dy = \Big|_0^1 \tfrac12 y^2 e^y - \int y e^y\,dy$ $= \tfrac12 e - \left[\Big|_0^1 ye^y - \int_0^1 e^y\,dy\right] = \tfrac12 e - e + \Big|_0^1 e^y = -\tfrac12 e + e - 1 = \tfrac12 e - 1$ Order of integration relevant!~~

Opposite: We then get $\int_0^1\left[\int_0^1 xe^y\,dx - \int_0^y xe^y\,dx\right]dy = \int_0^1\left(\int_y^1 xe^y\,dx\right)dy$

Order of integration not relevant!

$$= \int_0^1\left[\Big|_y^1 \tfrac12 x^2 e^y\right]dy = \int_0^1 \tfrac12 e^y - \tfrac12 y^2 e^y\,dy = \tfrac12(e-1)-\tfrac12\left[\Big|_0^1 y^2 e^y - \int_0^1 2ye^y\,dy\right]$$

$$= \tfrac12(e-1)-\tfrac12\left[e-2\left(\Big|_0^1 ye^y - \int_0^1 e^y\,dy\right)\right] = -\tfrac12 + e - \Big|_0^1 e^y = \tfrac12$$

Same number from both orders of integratio

$\boxed{\text{Problem 5-04}}$

a) $\dot{x} + 4x = 4e^{-2t} \quad x(0) = 1$

$$\dot{x}e^{4t} + 4xe^{4t} = 4e^{2t} \qquad \frac{d}{dt}\left(xe^{4t}\right) = 4e^{2t} \qquad xe^{4t} = 4\int e^{2t}\,dt = 2e^{2t} + C$$

$$x = 2e^{-2t} + Ce^{-4t} \qquad x(0) = 2 + C = 1 \Rightarrow C = -1 \quad \Rightarrow \underline{\underline{x = 2e^{-2t} - e^{-4t}}}$$

b) $\dot{k} = s(a + \sigma k)\sqrt{t+1} \quad k(0) = k_0$

$$\frac{dk}{dt} = s(a+\sigma k)\sqrt{t+1} \qquad \frac{1}{(a+\sigma k)}\,dk = s\sqrt{t+1}\,dt \qquad \int \frac{1}{a+\sigma k}\,dk = \int s\sqrt{t+1}\,dt$$

$$\frac{\ln(a+\sigma k)}{\sigma} = s\,\frac{2}{3}(t+1)^{\frac{3}{2}} + C \qquad \ln(a+\sigma k) = \sigma s\,\frac{2}{3}(t+1)^{\frac{3}{2}} + \sigma C$$

$$a + \sigma k = e^{\sigma s\frac{2}{3}(t+1)^{\frac{3}{2}} + \sigma C} \qquad \underline{k = \frac{e^{\sigma s\frac{2}{3}(t+1)^{\frac{3}{2}} + \sigma C} - a}{\sigma}}$$

$$k(0) = \frac{e^{\frac{2}{3}\sigma s + \sigma C} - a}{\sigma} = k_0 \qquad e^{\frac{2}{3}\sigma s + \sigma C} = \sigma k_0 + a \qquad \underline{e^{\sigma C} = \frac{\sigma k_0 + a}{e^{\frac{2}{3}\sigma s}}}$$

$$\Rightarrow \underline{\underline{k = \frac{\frac{\sigma k_0 + a}{e^{\frac{2}{3}\sigma s}}\, e^{\sigma s\frac{2}{3}(t+1)^{\frac{3}{2}}} - a}{\sigma}}}$$

Problem 5-06

$\dot{K} = \gamma Q \quad Q = K^{\gamma} L \quad \dot{L} = \beta \quad \gamma, \beta > 0 \quad 0 < \gamma < 1$

a) $\dot{K} = \gamma Q = \gamma K^{\gamma} L \quad \dot{L} = \beta \quad \frac{dL}{dt} = \beta \quad L(t) = \beta t + C$

$\Rightarrow \underline{\underline{\dot{K} = \gamma K^{\gamma}(\beta t + C)}}$

b) $L(0) = L_0 \Rightarrow C = L_0 \Rightarrow \underline{L = \beta t + L_0}$

$\Rightarrow \dot{K} = \gamma K^{\gamma}(\beta t + L_0) \Rightarrow \int \frac{1}{\gamma K^{\gamma}} dK = \int \beta t + L_0 \, dt$

$\frac{1}{\gamma} K^{1-\gamma} \frac{1}{1-\gamma} = \frac{1}{2}\beta t^2 + L_0 t + C \quad \underline{K = \left[\frac{1}{2}\beta t^2 + L_0 t + C\right]^{\frac{1}{1-\gamma}} (1-\gamma)^{\frac{1}{1-\gamma}} \gamma^{\frac{1}{1-\gamma}}}$

$K(0) = [C(1-\gamma)\gamma]^{\frac{1}{1-\gamma}} = K_0 \quad \underline{C = K_0^{1-\gamma} \frac{1}{(1-\gamma)\gamma}}$

$\Rightarrow \underline{\underline{K = \left[\frac{1}{2}\beta t^2 + L_0 t + \frac{K_0^{1-\gamma}}{(1-\gamma)\gamma}\right]^{\frac{1}{1-\gamma}} [(1-\gamma)\gamma]^{\frac{1}{1-\gamma}}}}$

Problem 5-11

$e^{2t}\dot{x} + e^{2t}(2-2t)x = \frac{e^{t^2+t}}{\sqrt{1+e^t}} \qquad e^{2t-t^2}\dot{x} + e^{2t-t^2}(2-2t)x = \frac{e^t}{\sqrt{1+e^t}}$

$\frac{d}{dt}(e^{2t-t^2}x) = \frac{e^t}{\sqrt{1+e^t}} \qquad e^{2t-t^2}x = \int \frac{e^t}{\sqrt{1+e^t}} dt \qquad u = e^t \quad \frac{du}{dt} = e^t \quad dt = \frac{1}{e^t} du$

$\Rightarrow e^{2t-t^2}x = \int \frac{1}{\sqrt{1+u}} du = 2(1+u)^{\frac{1}{2}} + C = 2(1+e^t)^{\frac{1}{2}} + C$

$\underline{x = 2\sqrt{1+e^t}\, e^{t^2-2t} + C e^{t^2-2t}} \qquad x(-1) = 2\sqrt{1+e^{-1}}\, e^3 + C e^3 = 0 \qquad \underline{C = -2\sqrt{1+e^{-1}}}$

$\Rightarrow x = 2\sqrt{1+e^t}\, e^{t^2-2t} - 2\sqrt{1+e^{-1}}\, e^{t^2-2t} \qquad \underline{\underline{x = 2e^{t^2-2t}\left[\sqrt{1+e^t} - \sqrt{1+e^{-1}}\right]}}$

Problem 5-14

$\dot{x} = x^3 + 3x^2 - 2$ $\quad x = y + a \Rightarrow \dot{x} = \dot{y}$, $x^3 = (y+a)(y^2+2ya+a^2) = y^3 + 3ay^2 + 3a^2y + a^3$

$x^2 = y^2 + 2ya + a^2$

$\Rightarrow \dot{y} = y^3 + 3(a+1)y^2 + 3a[a+2]y + a^2[3+a] - 2$

$a = -1$ gives $\underline{\dot{y} + 3y = y^3}$ Bernoulli's equation

$z = y^{1-3}$ $\underline{z = y^{-2}}$ $-\frac{1}{2}\dot{z} + 3z = 1 \Rightarrow \dot{z} - 6z = -2$

$\frac{d}{dt}(ze^{-6t}) = -2e^{-6t}$ $\quad ze^{-6t} = \frac{1}{3}e^{-6t} + C$ $\quad \underline{z = \frac{1}{3} + Ce^{6t}}$

$\Rightarrow y = (-2te^{6t} + Ce^{6t})^{-\frac{1}{2}} = \underline{\underline{(C-2t)^{-\frac{1}{2}}e^{-3t}}}$ $\quad y = \left(\frac{1}{3} + Ce^{6t}\right)^{-\frac{1}{2}}$

$$\underline{\underline{x = \left(\frac{1}{3} + Ce^{6t}\right)^{-\frac{1}{2}} - 1}}$$

Problem 6-08

i) $g''(t) = -\frac{1}{4}g'(t)$ $\quad$ Define $y = g'(t)$

The equation then says $\dot{y} = -\frac{1}{4}y \Rightarrow \dot{y} + \frac{1}{4}y = 0$ $\quad \frac{d}{dt}(ye^{\frac{1}{4}t}) = 0$ $\quad y = Ce^{-\frac{1}{4}t}$

$\Rightarrow \underline{\underline{g(t) = -4Ce^{-\frac{1}{4}t} + D}}$

ii) $g''(t) = -\frac{2}{t+1}g'(t)$ $\quad$ Define $y = g'(t)$

The equation then says $\dot{y} = -\frac{2}{t+1}y$ $\quad \dot{y} + \frac{2}{t+1}y = 0$

$\frac{d}{dt}\left(ye^{2\ln(t+1)}\right) = 0$ $\quad y(t+1)^2 = C$ $\quad y = \frac{C}{(t+1)^2}$

$\Rightarrow \underline{\underline{g(t) = -\frac{C}{t+1} + D}}$

Problem 6-10a

a) $\frac{1}{2}\sigma^2x^2V''(x) + \mu xV'(x) - \rho V(x) = w - x$ $\quad V(x) = Ax^a + Bx^b$

$V'(x) = aAx^{a-1} + bBx^{b-1}$ $\quad V''(x) = a(a-1)Ax^{a-2} + b(b-1)Bx^{b-2}$

~~homogeneous~~

$\Rightarrow \frac{1}{2}\sigma^2[a(a-1)Ax^a + b(b-1)x^b] + \mu[aAx^a + bBx^b] - \rho[Ax^a + Bx^b] = 0$ $\quad (*)$

$\Rightarrow \frac{1}{2}\sigma^2a(a-1) + \mu a - \rho = 0$ $\quad$ Gives a and b solutions to this equation, and $V(x) = Ax^a + Bx^b$,