# ECON 4140 - Seminar 7

## Problem 4-08

$$\int_0^1\int_0^{\frac{\pi}{2}} xy^2\cos(x^2y)\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_0^1 xy^2\cos(x^2y)\,dx\,dy$$

$u = x^2y \quad \frac{du}{dx} = 2xy \quad dx = \frac{1}{2xy}du$

$$= \int_0^{\frac{\pi}{2}}\int_0^{y} \frac{1}{2}y\cos(u)\,du\,dy = \int\left(\Big|_0 \frac{1}{2}y\sin(u)\right)dy = \int_0^{\frac{\pi}{2}} \frac{1}{2}y\sin(y)\,dy$$

$$= \frac{1}{2}\Bigg[\underbrace{\Big|_0^{\frac{\pi}{2}} -y\cos(y)}_{=0} + \int_0^{\frac{\pi}{2}}\cos(y)\,dy\Bigg] =$$

$$= -\frac{1}{2}$$

## Problem 4-09

$$\int_\pi^{2\pi}\int_0^{\pi}\frac{x}{y^3}\cos\left(\frac{x^2}{y}\right)dx\,dy$$

$u = \frac{x^2}{y} \quad \frac{du}{dx} = 2\frac{x}{y} \quad dx = \frac{1}{2}\frac{y}{x}du$

$$= \int_\pi^{2\pi}\int_0^{\frac{\pi^2}{y}}\frac{1}{2}\frac{1}{y^2}\cos(u)\,du\,dy = \int_\pi^{2\pi}\left(\Big|_0^{\frac{\pi^2}{y}}\frac{1}{2}\frac{1}{y^2}\sin(u)\right)dy = \int_\pi^{2\pi}\frac{1}{2}\frac{1}{y^2}\sin\left(\frac{\pi^2}{y}\right)dy$$

$v = \frac{\pi^2}{y}$

$\frac{dv}{dy} = -\frac{\pi^2}{y^2}$

$dy = -\frac{y^2}{\pi^2}dv$

$$= \int_\pi^{\frac{\pi}{2}} -\frac{1}{2\pi^2}\sin(v)\,dv = \frac{1}{2\pi^2}\int_{\frac{\pi}{2}}^{\pi}\sin(v)\,dv = \frac{1}{2\pi^2}\Big|_{\frac{\pi}{2}}^{\pi}\left(-\cos(v)\right) = \frac{1}{2\pi^2}$$

$\boxed{\text{Problem 6-02}}$

a) $\ddot{x} + \frac{7}{2}\dot{x} - 2x = 0$  $\frac{1}{4}\left(\frac{7}{2}\right)^2 - (-2) = \frac{49}{16} + 2 > 0$ ⇒ Are in case (I)

⇒ $x = Ae^{r_1 t} + Be^{r_2 t}$ where $r_{1,2} = -\frac{7}{4} \pm \sqrt{\frac{49}{16} + 2}$ ⇒ $\boxed{r_1 = \frac{1}{2} \quad r_2 = -4}$

b) $f(t) = t + \sin t$ ⇒ $u^* = At + B + C\sin t + D\cos t$

$\dot{u}^* = A + C\cos t - D\sin t$  $\ddot{u}^* = -C\sin t - D\cos t$

⇒ $-C\sin t - D\cos t + \frac{7}{2}(A + C\cos t - D\sin t) - 2(At + B + C\sin t + D\cos t) = t + \sin t$ ~~$At + B + C\sin t + D\cos t$~~

⇒ $\frac{7}{2}A - 2B = 0$  $B = \frac{7}{4}A$  $-2A = 1$ ⇒ $\boxed{A = -\frac{1}{2}}$ ⇒ $\boxed{B = -\frac{7}{8}}$

$-D + \frac{7}{2}C - 2D = 0$ ⇒ $D = \frac{7}{6}C$

$-C - \frac{7}{2}D - 2C = 1$  ⇒ $-3C - \frac{7}{2}\frac{7}{6}C = 1$  $C\left[3 + \frac{49}{12}\right] = -1$  $\boxed{C = -\frac{12}{61}}$

$\boxed{D = -\frac{14}{61}}$  $u^* = -\frac{1}{2}t - \frac{7}{8} - \frac{12}{61}\sin t - \frac{14}{61}\cos t$

⇒ $x = Ae^{\frac{1}{2}t} + Be^{-4t} - \frac{1}{2}t - \frac{7}{8} - \frac{12}{61}\sin t - \frac{14}{61}\cos t$

**Problem 6-11**

$$\ddot{x} - 2(k-1)\dot{x} + (k^2-4)x = 2e^{(4-k)t}$$

a) $\ddot{x} - 2(k-1)\dot{x} + (k^2-4)x = 0$

$$\frac{1}{4}(-2(k-1))^2 - (k^2-4) = (k-1)^2 - k^2 + 4 = \underline{5-2k} \begin{cases} >0 & \text{if } k<\frac{5}{2} \\ =0 & \text{if } k=\frac{5}{2} \\ <0 & \text{if } k>\frac{5}{2} \end{cases}$$

• $k<\frac{5}{2} \Rightarrow \underline{x = Ae^{r_1 t} + Be^{r_2 t}, \; r_{1,2} = (k-1) \pm \sqrt{5-2k}}$

• $k=\frac{5}{2} \Rightarrow \underline{x = (A+Bt)e^{rt}, \; r = k-1}$

• $k<\frac{5}{2} \Rightarrow \underline{x = e^{(k-1)t}(A\cos pt + B\sin pt), \; p = \sqrt{2k-5}}$

b) $f(t) = 2e^{(4-k)t} \Rightarrow x^* = \dfrac{2}{(4-k)^2 - 2(k-1)(4-k) + (k^2-4)} e^{(4-k)t} = \underline{\dfrac{2e^{(4-k)t}}{4k^2 - 18k + 20}}$

c) $k=3 \Rightarrow \underline{x = e^{2t}(A\cos t + B\sin t) + e^t}$

Passes through origin $\Rightarrow x(0) = 0 \Rightarrow A + 1 = 0 \Rightarrow \underline{A = -1}$

Tangent to t-axis at origin $\Rightarrow \dot{x}(0) = 0$

$\dot{x} = 2e^{2t}(B\sin t - \cos t) + e^{2t}(B\cos t + \sin t) + e^t$

$\dot{x}(0) = 2(-1) + B + 1 = 0 \Rightarrow \underline{B = 1}$

$\Rightarrow \underline{x = e^{2t}(\sin t - \cos t) + e^t}$

$\dot{x} = 2e^{2t}(\sin t - \cos t) + e^{2t}(\cos t + \sin t) + e^t = e^{2t}(3\sin t - \cos t) + e^t$

$\ddot{x} = 2e^{2t}(3\sin t - \cos t) + e^{2t}(3\cos t + \sin t) + e^t = e^{2t}(7\sin t + \cos t) + e^t$

$\ddot{x}(0) = 1 + 1 = 2 > 0 \Rightarrow$ <u>$(x,t) = (0,0)$ is local minimum, and thus local extrema</u>

$\boxed{\text{Problem 6-15}}$

a) $2\ddot{x}+8\dot{x}+26x=e^{2t} \Rightarrow \ddot{x}+4\dot{x}+13x=\frac{1}{2}e^{2t}$

Homogeneous: $\ddot{x}+4\dot{x}+13x=0 \quad \frac{1}{4}4^2-13=4-13<0$

$\Rightarrow$ ~~$e^{-2t}(A\cos 3t$~~ $\underline{x=e^{-2t}(A\cos(3t)+B\sin(3t))}$

$$u^*=\frac{\frac{1}{2}}{4+\frac{1}{2}\cdot 4+13}e^{2t}=\frac{e^{2t}}{8+4+26}=\underline{\frac{e^{2t}}{38}}$$

$$\Rightarrow \underline{\underline{x=e^{-2t}(A\cos(3t)+B\sin(3t))+\frac{e^{2t}}{38}}}$$

b) $e^{2t}$ replaced by $\sin 3t \Rightarrow \ddot{x}+4\dot{x}+13x=\frac{1}{2}\sin 3t \Rightarrow f(t)=\frac{1}{2}\sin 3t$

$\Rightarrow u^*=A\sin 3t+B\cos 3t$, where $A$ and $B$ is found by fitting the relevant coefficients in the equation $\ddot{u}^*+4\dot{u}^*+13u^*=\frac{1}{2}\sin 3t$.

$\boxed{\text{Problem 6-10}}\boxed{\text{b\&c}}$

b) $\frac{1}{2}\sigma^2x^2V''(x)+\mu xV'(x)-\rho V(x)=w-x$

A particular solution: $u^*=Ax+B \quad \dot{u}^*=A \quad \ddot{u}^*=0$

$\Rightarrow \mu xA-\rho(Ax+B)=w-x \Rightarrow \mu A-\rho A=-1 \quad \boxed{A=\frac{1}{\rho-\mu}}$

$-\rho B=w \quad \boxed{B=-\frac{w}{\rho}} \quad \Rightarrow \underline{\underline{u^*=\frac{x}{\rho-\mu}-\frac{w}{\rho}}}$

c) Here, $\sigma=\sqrt{2}$, $\mu=1$, $\rho=4$, $w=10$.

$\Rightarrow$ ~~$V(x)=Ax^a+Bx^b+\frac{1}{3}x-\frac{5}{2}$~~

where $a,b$ is solutions to $a(a-1)+a-4=0 \quad a^2-4=0 \Rightarrow a=\pm 2$

$$\Rightarrow \underline{\underline{V(x)=Ax^2+Bx^{-2}+\frac{1}{3}x-\frac{5}{2}}}$$

# Problem 6-13

$t^2\ddot{x} + t\dot{x} - x = 0 \quad , t > 0$

a) (i) Without substitution:

Characteristic equation $r^2 - 1 = 0 \Rightarrow r^2 = 1 \Rightarrow \underline{r_{1,2} = \pm 1}$

$\Rightarrow \underline{\underline{x = At + Bt^{-1}}}$

(ii) With substitution:

$z = tx \quad \dot{z} = x + t\dot{x} \quad \ddot{z} = \dot{x} + \dot{x} + t\ddot{x} = 2\dot{x} + t\ddot{x}$

$\Rightarrow t^2\ddot{x} + t\dot{x} - x = t\ddot{z} - \dot{z}$

So get $\ddot{z} - \frac{1}{t}\dot{z} = 0 \quad$ Let $y = \dot{z}$

$\Rightarrow \dot{y} - \frac{1}{t}y = 0 \quad \frac{1}{y}dy = \frac{1}{t}dt \quad \ln y = \ln t + C \quad y = te^C \quad \underline{y = tD}$

$\Rightarrow \dot{z} = tD \quad z = \frac{1}{2}t^2 D + E \quad z = D_2 t^2 + E = tx$

$\Rightarrow \underline{\underline{x = D_2 t + Et^{-1}}}$

b) $x(1) = A + B = 1 \quad \underline{B = 1 - A}$

$\dot{x} = A - Bt^{-2} \quad \dot{x}(1) = A - B = A + A - 1 = 1 \Rightarrow \underline{A = 1} \Rightarrow \underline{B = 0}$

$\Rightarrow \underline{\underline{x = t}}$

# Problem 6-12

$$\dot{p}(t) = \beta \int_{-\infty}^{t} [a+c-(b+d)p(\tau)]\, e^{-\gamma(t-\tau)}\, d\tau$$

a) $\ddot{p} = \beta\left[a+c-(b+d)p(t) - \gamma \int_{-\infty}^{t} (a+c-(b+d)p(\tau))\, e^{-\gamma(t-\tau)}\, d\tau\right]$

$$= \beta(a+c) - \beta(b+d)p - \gamma\dot{p}$$

$\Rightarrow$ $\underline{\ddot{p} + \gamma\dot{p} + \beta(b+d)p = \beta(a+c)}$

b) $\dot{p} = 0$ and $\ddot{p} = 0$ $\Rightarrow$ $\underline{p^* = \dfrac{a+c}{b+d}}$

General solution : $\frac{1}{4}\gamma^2 - \beta(b+d) \begin{cases} >0 & \text{case I} \\ =0 & \text{case II} \\ <0 & \text{case III} \end{cases}$

Case I : $p = Ae^{r_1 t} + Be^{r_2 t}$ where $r_{1,2} = -\frac{1}{2}\gamma \pm \sqrt{\frac{1}{4}\gamma^2 - \beta(b+d)}$

Case II : $p = (A+Bt)e^{-\frac{1}{2}\gamma t}$

Case III : $p = e^{-\frac{1}{2}\gamma t}(A\cos\theta t + B\sin\theta t)$ where $\theta = \sqrt{\beta(b+d) - \frac{1}{4}\gamma^2}$

To these general solution is $p + v^*$, when $v^* = p^*$ $\Rightarrow$ $p + \frac{a+c}{b+d}$

c) Since $\gamma > 0$, $-\frac{1}{2}\gamma < 0$, and also note that both $r_1 < 0$ and $r_2 < 0$

Thus, case I $\lim_{t\to\infty}(p + p^*) = p^*$ since $\lim_{t\to\infty} p = 0$

case II $\lim_{t\to\infty} (A+Bt)e^{-\frac{1}{2}\gamma t} + p^* = p^*$ (even though $Bt \to \infty$, the exponent will dominate!)

case III $\lim_{t\to\infty} (p + p^*) = p^*$ ~~fluctuate in the interval~~

$\Rightarrow$ Stable in all cases

In the case $\frac{1}{4}\gamma^2 - \beta(b+d) < 0$ we have the fluctuating case due to sin, cos functions.