# ECON 4140 - Seminar 8

## Problem 7-01

$$\dot{x} = x + y + t \quad (1)$$
$$\dot{y} = -x + 2y \quad (2)$$

$$\cancel{\ddot{x} = \dot{x} + \dot{y} + 1 = \dot{x} - x + 2y + 1}$$

(2) gives $x = 2y - \dot{y} \Rightarrow \dot{x} = 2\dot{y} - \ddot{y}$

Thus, (1) becomes $2\dot{y} - \ddot{y} = 2y - \dot{y} + y + t$ i.e. $\underline{\ddot{y} - 3\dot{y} + 3y = -t}$ (*)

$$\frac{1}{4}(-3)^2 - 3 = \frac{9}{4} - 3 = \frac{9-12}{4} < 0$$

$\Rightarrow x = e^{\frac{3}{2}t}(A\cos pt + B\sin pt)$ where $p = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$$\underline{x = e^{\frac{3}{2}t}\left(A\cos\left(\tfrac{\sqrt{3}}{2}t\right) + B\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right)}$$

$u^* = At + B \quad \dot{u}^* = A \quad \ddot{u}^* = 0$

$\Rightarrow$ inserted in (*) gives $-3A + 3At + 3B = -t \Rightarrow 3A = -1$ $\boxed{A = -\frac{1}{3}}$

$-3A + 3B = 0 \qquad 3B = -1$ $\boxed{B = -\frac{1}{3}}$ $\Rightarrow \underline{u^* = -\frac{1}{3}t - \frac{1}{3}}$

$\Rightarrow$ General solution of $x$: $\underline{\underline{x = e^{\frac{3}{2}t}\left(A\cos\left(\tfrac{\sqrt{3}}{2}t\right) + B\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right) - \frac{1}{3}t - \frac{1}{3}}}$

$\Rightarrow \dot{x} = \frac{3}{2}e^{\frac{3}{2}t}\left(A\cos\left(\tfrac{\sqrt{3}}{2}t\right) + B\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right) + e^{\frac{3}{2}t}\left(-\frac{\sqrt{3}}{2}A\sin\left(\tfrac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}B\cos\left(\tfrac{\sqrt{3}}{2}t\right)\right) - \frac{1}{3}$

$\Rightarrow y = \dot{x} - x - t = \underline{\underline{\frac{1}{2}e^{\frac{3}{2}t}\left[A\cos\left(\tfrac{\sqrt{3}}{2}t\right) + B\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right] + e^{\frac{3}{2}t}\left[\frac{\sqrt{3}}{2}B\cos\left(\tfrac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}A\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right] - \frac{2}{3}t}}$

Problem 7-04 a & b

$\dot{x} = y - x^2 - xy = f(x,y)$
$\dot{y} = x - y^2 - xy = g(x,y)$

a) $\dot{x} = 0 \Rightarrow y - x^2 - xy = 0 \Rightarrow xy = y - x^2$
$\dot{y} = 0 \Rightarrow x - y^2 - xy = 0$

$x - y^2 - y + x^2 = 0 \qquad x + x^2 = y + y^2 \Rightarrow \underline{x = y}$

$\Rightarrow x - 2x^2 = 0 \qquad x^2 - \frac{1}{2}x = 0 \qquad x(x - \frac{1}{2}) = 0 \qquad x = 0$ or $x = \frac{1}{2}$

$\Rightarrow \boxed{(x,y) = (0,0) \text{ or } (x,y) = (\tfrac{1}{2}, \tfrac{1}{2})}$

~~$0 = y - x^2 - xy$ partial derivative w.r.t. x $f(x,y)$ $0 = -2x - y$~~

$f_1' = -2x - y \qquad f_2' = 1 - x$

$g_1' = 1 - y \qquad g_2' = -2y - x$

$A(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow tr(A(0,0)) = 0 \;, |A(0,0)| = -1 < 0$

Since the determinant is negative, $\underline{(0,0) \text{ is saddle point}}$

$A(\frac{1}{2},\frac{1}{2}) = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \Rightarrow tr(A(\frac{1}{2},\frac{1}{2})) = -3 < 0 \qquad |A(\frac{1}{2},\frac{1}{2})| = \frac{9}{4} - \frac{1}{4} = 2 > 0$

Since determinant positive and trace negative, $\underline{(\tfrac{1}{2},\tfrac{1}{2}) \text{ is asymptotically stable}}$

b) $z = x + y \qquad \dot{z} = \dot{x} + \dot{y} = y - x^2 - xy + x - y^2 - xy = x + y - (x^2 + 2xy + y^2) = z - z^2$

$\Rightarrow \underline{\dot{z} = z - z^2}$ ~~Bernoulli~~ i.e. $\underline{\dot{z} - z = -z^2}$ Bernoulli form

$\Rightarrow w = z^{-1} \qquad 0 - \dot{w} - w = -1 \qquad \dot{w} + w = 1 \qquad \frac{d}{dt}(e^t w) = e^t \qquad e^t w = e^t + C$

~~$w = e^{-t} + Ce^{-t} \Rightarrow z = \frac{1}{1 + Ce^{-t}}$~~ $\quad w = 1 + Ce^{-t} \Rightarrow z = \frac{1}{1 + Ce^{-t}} \qquad \underline{z = \frac{e^t}{e^t + C}}$

## $\boxed{\text{Exercise by Fromstad}}$

$\dot{x} = 1 - e^{x-y} \qquad \dot{y} = -y$

$\dot{x} = 0 \Rightarrow 1 - e^{x-y} = 0$

$\dot{y} = 0 \Rightarrow -y = 0 \Rightarrow \boxed{y = 0} \Rightarrow 1 - e^{x} = 0 \Rightarrow e^{x} = 1 \Rightarrow \boxed{x = 0}$

$\Rightarrow$ $(x,y) = (0,0)$ is equilibrium point

## $\boxed{\text{Exercise 7-05}}$

$\dot{x} = \underbrace{\tfrac{1}{2}x^3 - y}_{f(x,y)} \qquad \dot{y} = \underbrace{2x - y}_{g(x,y)}$

a) $\dot{x} = 0 \Rightarrow \tfrac{1}{2}x^3 - y = 0 \quad y = \tfrac{1}{2}x^3$

$\dot{y} = 0 \Rightarrow 2x - y = 0 \quad 2x = \tfrac{1}{2}x^3$ $\quad x = 0$ or $x^2 = 4$ is $x = \pm 2$

$\Rightarrow y = 0$ or $y = \pm 4$

$\boxed{(x,y) = (0,0), (2,4), (-2,-4)}$ Equilibrium points.

$f'_x = \tfrac{3}{2}x^2 \quad f'_y = -1 \qquad A = \begin{pmatrix} \tfrac{3}{2}x^2 & -1 \\ 2 & -1 \end{pmatrix} \qquad tr(A) = \tfrac{3}{2}x^2 - 1$

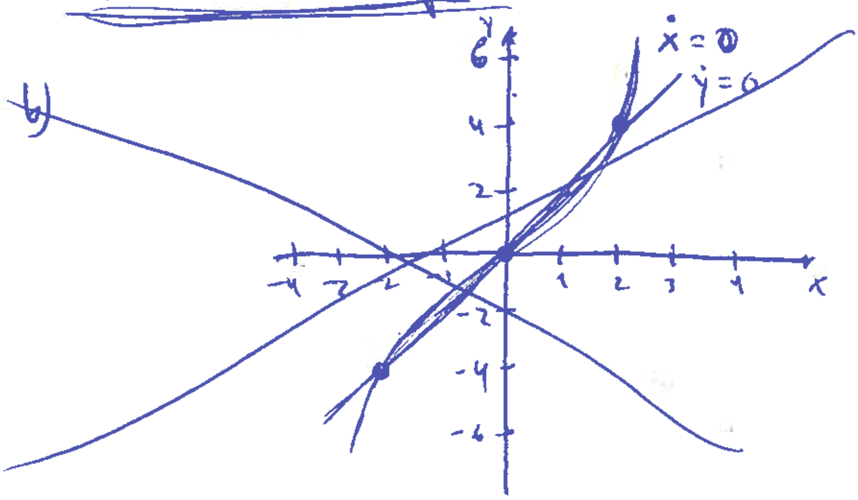
$g'_x = 2 \quad g'_y = -1 \qquad |A| = -\tfrac{3}{2}x^2 + 2$

$(0,0)$ : $tr(A) < 0 \quad |A| > 0 \Rightarrow$ Asymptotically stable

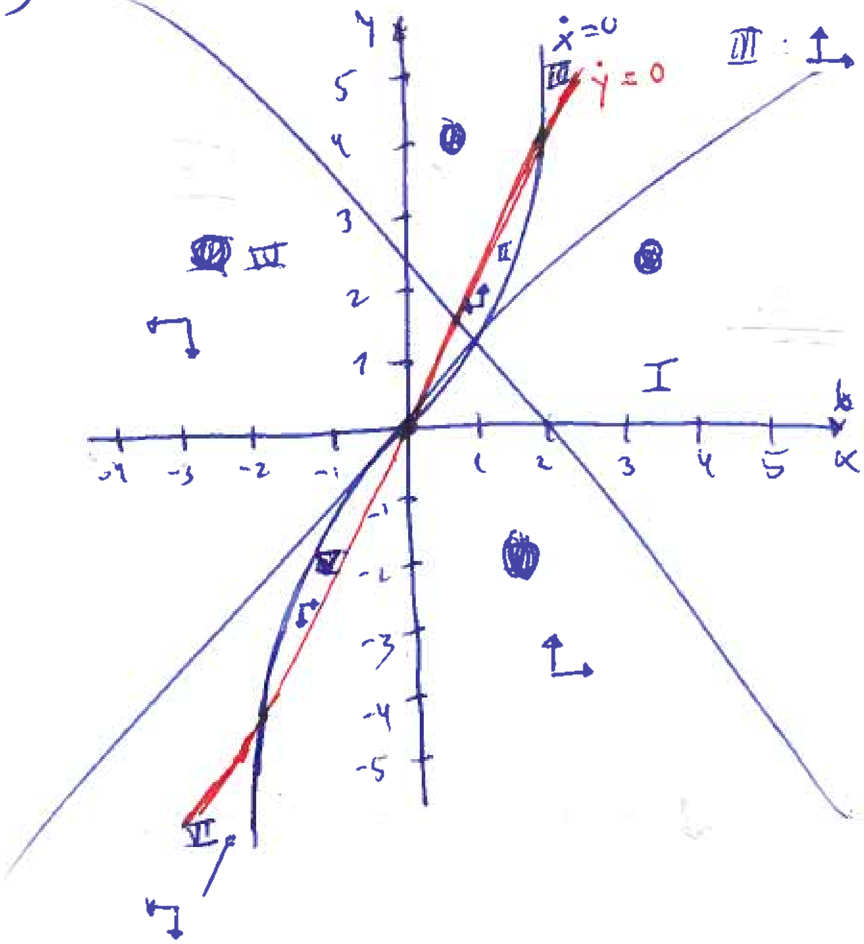
$(2,4)$ : $tr(A) > 0 \quad |A| < 0 \Rightarrow$ Saddle path

$(-2,-4)$ Also saddle path

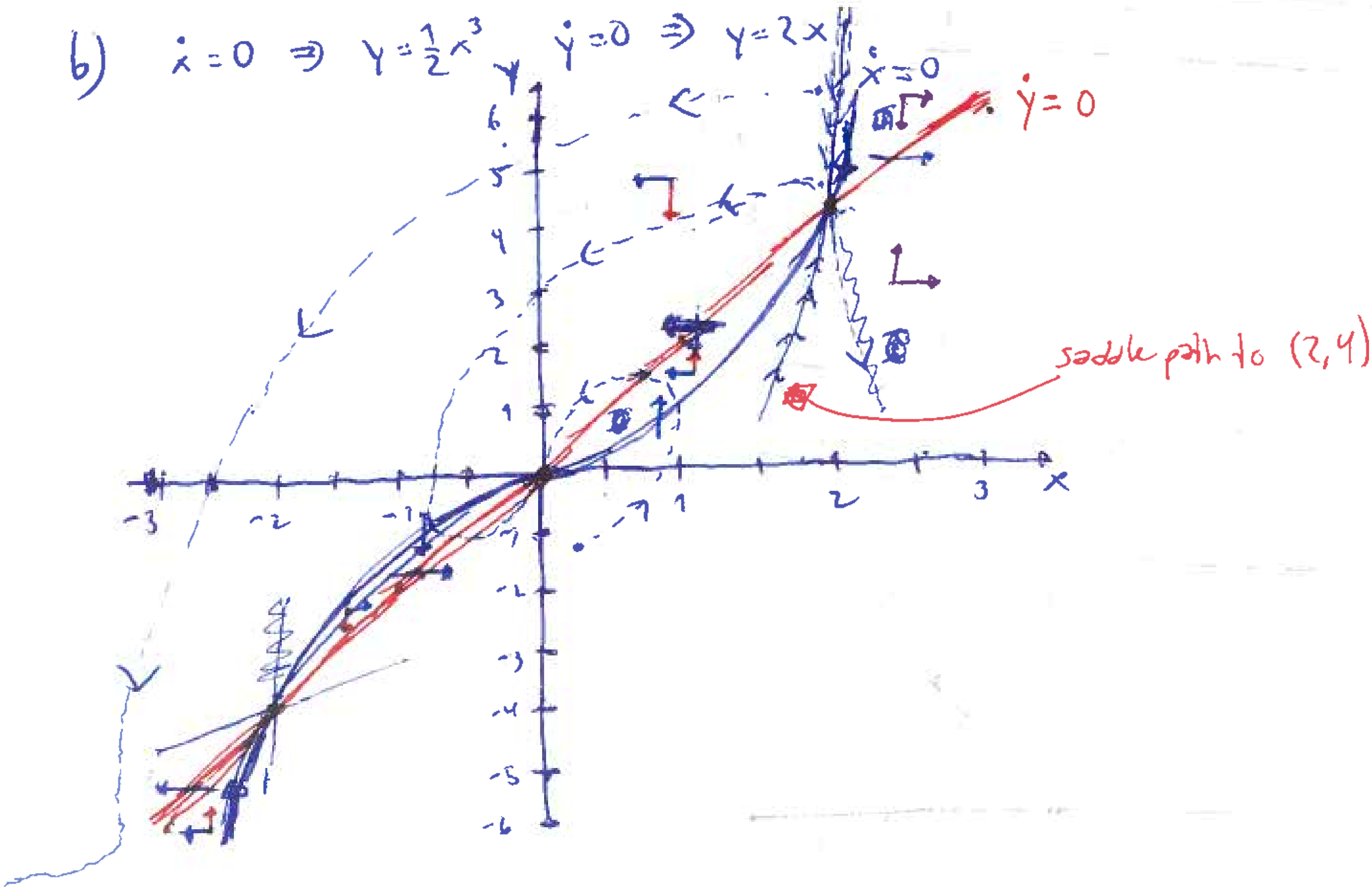
b)



b) $\dot{x} = 0 \Rightarrow y = \frac{1}{2}x^3$ $\quad \dot{y} = 0 \Rightarrow y = 2x$




b) $\dot{x} = 0 \Rightarrow y = \frac{1}{2}x^3$ $\quad \dot{y} = 0 \Rightarrow y = 2x$



c) This is a function $f(x)$ such that $f(2)=4$

and • $\frac{d}{dt}(f(x)) = \underline{f'(x)\,\dot{x} = 2x - f(x)}$ (1)

• $\underline{\dot{x} = \frac{1}{2}x^3 - f(x)}$ (2)

(2) in (1) gives $f'(x)\left(\frac{1}{2}x^3 - f(x)\right) = 2x - f(x)$ $\Rightarrow$ $\underline{f'(x) = \frac{2x - f(x)}{\frac{1}{2}x^3 - f(x)}}$

Thus $f'(2) = \frac{"0"}{0}$, use L'Hôpital's rule

gets : $\cancel{f'(2) = \frac{2-f'(2)}{...}}$ $f'(2) = \lim_{x\to 2} \frac{2 - f'(x)}{\frac{3}{2}x^2 - f'(x)} = \frac{2 - f'(2)}{6 - f'(2)}$

This gives the equation $z(6-z) = 2 - z$ ie $\underline{z^2 - 7z + 2 = 0}$

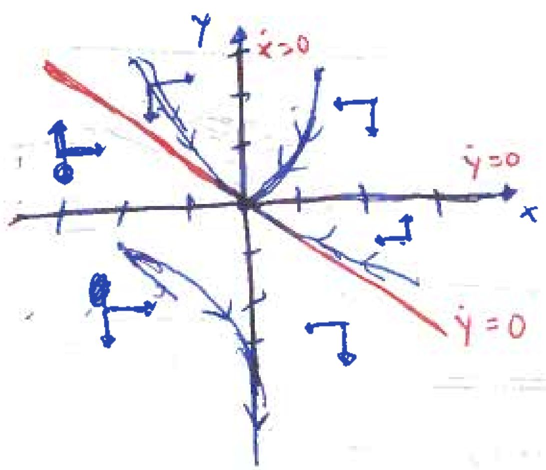
Solution: $z = \frac{7 \pm \sqrt{49-8}}{2} = \frac{7 \pm \sqrt{41}}{2}$ The slope will be positive (see drawing)

thus $\underline{\underline{f'(2) = \frac{7 + \sqrt{41}}{2}}}$

$\boxed{\text{Exercise 7.06}}$

$\dot{x} = -x = f(x,y)$
$\dot{y} = -xy - y^2 = g(x,y)$

a) $\dot{x} = 0 \Rightarrow \underline{x = 0}$ $\dot{y} = 0 \Rightarrow y^2 = -xy$ i.e. $\underline{y = 0}$ or $\underline{y = -x}$



b) $(0,0)$ only equilibrium point

$f'_1 = -1 \qquad f'_2 = 0$

$g'_1 = -y \qquad g'_2 = -x - 2y$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} \operatorname{trace}(A) < 0 \\ |A| = 0 \end{array}$$

No result from this, but from the phase diagram one observes that $(0,0)$ is <u>not stable</u>.

c) $\dot{x} = -x \qquad \dot{x} + x = 0 \qquad \frac{d}{dt}(x e^{t}) = 0 \qquad x e^{t} = C \qquad x = C e^{-t}$

$x(0) = \underline{C = -1} \qquad \Rightarrow \underline{x = -e^{-t}}$

$\Rightarrow \dot{y} = e^{-t} y - y^2 \qquad \dot{y} - e^{-t} y = -y^2$ Bernoulli

$z = y^{-1} \qquad -\dot{z} - e^{-t} z = -1 \qquad \dot{z} + e^{-t} z = 1$

$\frac{d}{dt}\left(z e^{-e^{-t}}\right) = e^{-e^{-t}} \Rightarrow e^{-e^{-t}} z = \int e^{-e^{-s}} ds + C \qquad \underline{z = e^{e^{-t}} \int_0 e^{-e^{-s}} ds + e^{e^{-t}} C}$

$\Rightarrow y = \dfrac{1}{e^{e^{-t}} \int_0 e^{-e^{-s}} ds + e^{e^{-t}} C}$

$y(0) = \frac{1}{eC} = 1 \Rightarrow \boxed{C = \frac{1}{e}}$

$y = \dfrac{1}{e^{e^{-t}} \int_0 e^{-e^{-s}} ds + e^{e^{-t} - 1}}$

$\underline{\lim_{t \to \infty} x = 0} \qquad \underline{\lim_{t \to \infty} y = 0}$ (notice $e^{-e^{-s}} \to \infty$ as $s$ grows).

↳ This means the point is a point on a saddle path.

$\boxed{\text{Exam 2008, Problem 1}}$

a) $C_k = \begin{pmatrix} k & 2 & k \\ 2 & 3 & 0 \\ k & 0 & k \end{pmatrix}$ $\qquad D_k = \begin{pmatrix} k & 2 & k & k \\ 2 & 3 & 0 & k+1 \\ k & 0 & k & ke^k \end{pmatrix}$

$$|C_k| = k\begin{vmatrix} 3 & 0 \\ 0 & k \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ k & k \end{vmatrix} + k\begin{vmatrix} 2 & 3 \\ k & 0 \end{vmatrix} = 3k^2 - 4k - 3k^2 = -4k$$

$\boxed{k \neq 0 \Rightarrow \text{Rank}(C_k) = 3}$ $\Rightarrow$ $\boxed{\text{Rank}(D_k) = 3 \text{ for } k \neq 0}$

$\boxed{k = 0 \Rightarrow \text{Rank}(C_0) = 2}$

$k = 0 \Rightarrow D_0 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\Rightarrow$ $\boxed{\text{Rank}(D_0) = 2}$

<u>Has solution for all $k \neq 0$</u>

b) $A = C_0 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

i) $\begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \begin{pmatrix} 2y \\ 2x+3y \\ 0 \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$ $\quad \Rightarrow \underline{z = 0} \quad \underline{y = 2x}$

$2x + 6x = 8x$

$\Rightarrow \underline{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}$

ii) $\begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \lambda_2\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\lambda_2 \\ -\lambda_2 \\ 0 \end{pmatrix}$ $\quad \Rightarrow \underline{\lambda_2 = -1}$

iii) ~~trace A~~ $\text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 3$

$4 - 1 + \lambda_3 = 3 \qquad \underline{\lambda_3 = 0}$

d) i) $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y \\ 2x+3y \\ 0 \end{pmatrix}$

$\dot{z} = 0 \Rightarrow \underline{z = C}$ $\qquad \dot{x} = 2y \quad \underline{y = \frac{1}{2}\dot{x}} \quad \underline{\dot{y} = \frac{1}{2}\ddot{x}}$

$\dot{y} = 2x + 3y \quad \Rightarrow \frac{1}{2}\ddot{x} = 2x + \frac{3}{2}\dot{x} \Rightarrow \underline{\ddot{x} - 3\dot{x} - 4x = 0}$

$\frac{1}{4}9 + 4 > 0 \Rightarrow \underline{x = Ae^{4t} + Be^{-t}}$ $\qquad \underline{y = 2Ae^{4t} - \frac{1}{2}Be^{-t}}$

$\frac{9+16}{4} = \frac{25}{4}$

~~i) $x(0) = A + B = 0 \Rightarrow A = -B$~~

~~$y(0) = 2A - \frac{1}{2}B = -2B - \frac{1}{2}B = 0 \quad -4B - B = 0 \quad B = 0 \Rightarrow$~~

ii) We need $\lim_{t\to\infty} x = 0$ and $\lim_{t\to\infty} y = 0$

Only possible when $\boxed{A=0}$

Thus $x^* = Be^{-t}$ $y^* = -\frac{1}{2}Be^{-t}$ $\Rightarrow$ $\underline{\frac{x^*}{y^*} = -2}$ (Here, $B \neq 0$).

$\boxed{\text{Induction exercise}}$

a) $0 + 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

• Base case $0+1 = 1$ $\quad \frac{1(1+1)}{2} = \frac{2}{2} = 1$ ok for base case

• Assume true for $n$, is then true for $n+1$?

$0+1+\dots+n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$

Yes, is true for $n+1$ as well $\Rightarrow$ Hypothesis true

b) • $\frac{9-1}{8} = \frac{8}{8} = 1$ $\Rightarrow$ Possible for base case.

• Assume true for $n$, is true for $n+1$? This means $\frac{9^n - 1}{8} = k$ ← integer

~~$\frac{9^{n+1}}{8} - \frac{1}{8} = \frac{9 \cdot 9^n}{8} - \frac{9}{8} \cdot \frac{1}{8}$~~ or $9^n - 1 = 8k$ or $\underline{9^n = 8k+1}$

Then, $\frac{9^{n+1} - 1}{8} = \frac{9 \cdot 9^n - 1}{8} = \frac{9 \cdot (8k+1) - 1}{8} = \frac{9 \cdot 8k + 9 - 1}{8}$

$= 9k + \frac{9}{8} - \frac{1}{8} = 9k + \frac{8}{8} = 9k + 1$ which is integer since $k$ integer.

$\Rightarrow$ Hypothesis true

$\boxed{\text{Diff. equation problem}}$

- $\dot{x} = Ax + Ab \qquad x = -b \Rightarrow \dot{x} = -Ab + Ab = 0$ ok!

- $x(t)$ solution $\Rightarrow Ax + Ab = 0$ i.e. $\underline{Ax = -Ab}$

$y = x + b \Rightarrow \dot{y} = \dot{x} = \underline{Ax + Ab}$

$\dot{y} = Ay = \underline{Ax + Ab}$ <u>Yes, this is true</u>