ECON 4140 - Seminar 9

$\boxed{\text{Problem 10-3}}$

$$x_{t+2} + x_{t+1} - 6x_t = 5^t + t$$

$$1^2 + 6 \cdot 4 > 0 \quad \Rightarrow \quad x_t = Am_1^t + Bm_2^t \qquad m_{1,2} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{25} = -\tfrac{1}{2} \pm \tfrac{5}{2}$$

$$m_1 = 2 \qquad m_2 = -3$$

$$y_t = C5^t + Dt + E$$

$$C5^{t+2} + D(t+2) + E + C5^{t+1} + D(t+1) + E - 6(C5^t + Dt + E) = 5^t + t$$

$$5^t(25C + 5C - 6C) + t(D + D - 6D) + 2D + E + D + E - 6E = 5^t + t$$

$$24C = 1 \quad \underline{C = \tfrac{1}{24}} \qquad -4D = 1 \quad \underline{D = -\tfrac{1}{4}} \qquad -\tfrac{3}{4} - 4E = 0 \quad \underline{E = -\tfrac{3}{16}}$$

$$\Rightarrow \quad \underline{\underline{x_t = A2^t + B(-3)^t + \tfrac{1}{24}5^t - \tfrac{1}{4}t - \tfrac{3}{16}}}$$

$\boxed{10-04}$

$x_{t+2}+4x_{t+1}-12x_t = 7t^2+2t-6 \qquad x_0=-3 \quad x_1=9$

$4^2-4(-12)>0 \Rightarrow x_t = Am_1^t+Bm_2^t \quad m_{1,2} = -2 \pm \frac{1}{2}\sqrt{64} = -2\pm 4 \quad \underline{m_1=2} \ \underline{m_2=-6}$

$y_t = Ct^2+Dt+E$

$\cancel{C(t+2)^2+D(t+2)+E+C(t+1)^2+D(t+1)+E+Ct^2+Dt+E}$

$C(t+2)^2+D(t+2)+E+4[C(t+1)^2+D(t+1)+E]-12[Ct^2+Dt+E] = 7t^2+2t-6$

$t^2[C+4C-12C]+t[4C+D+8C+4D-12D]+[4C+2D+E+4C+4D+4E-12E] = 7t^2+2t-6$

$-7C=7 \quad \underline{C=-1} \qquad -12-7D=2 \quad \underline{D=-2}$

$-8-12-7E=-6 \quad \underline{E=-2}$

$\Rightarrow \underline{\underline{x_t = A2^t+B(-6)^t-t^2-2t-2}}$

$x_0 = A+B-2=-3 \quad \underline{A=-1-B}$

$x_1 = -(1+B)2-6B-1-2-2=9 \quad 8B=-16 \quad \underline{B=-2} \quad \underline{A=1}$

$\Rightarrow \underline{\underline{x_t = 2^t-2(-6)^t-t^2-2t-2}}$

$\boxed{10\text{-}05}$

a) $x_{t+2} - \frac{5}{2} x_{t+1} + x_t = 10 \cdot 3^t \qquad x_0 = 0 \quad x_1 = 2$

$\frac{25}{4} - 4 = \frac{25-16}{4} = \frac{9}{4} > 0 \Rightarrow m_{1,2} = \frac{5}{4} \pm \frac{1}{2}\sqrt{\frac{9}{4}} = \frac{5}{4} \pm \frac{3}{4} \quad \underline{m_1 = 2} \quad \underline{m_2 = \frac{1}{2}}$

$\underline{x_t = A 2^t + B 2^{-t}}$

$u_t = C 3^t$

$C 3^{t+2} - \frac{5}{2} C 3^{t+1} + C 3^t = 10 \cdot 3^t$

$3^t \left[9C - \frac{15}{2} C + C \right] = 10 \cdot 3^t \quad \frac{18-15+2}{2} C = 10 \quad C = \frac{20}{5} \quad \underline{C = 4}$

$\Rightarrow \underline{x_t = A 2^t + B 2^{-t} + 4 \cdot 3^t}$

$x_0 = A + B + 4 = 0 \quad A = -B - 4$

$x_1 = -2B - 8 + \frac{1}{2} B + 12 = 2 \quad \frac{3}{2} B = 2 \quad \underline{B = \frac{4}{3}} \quad A = -\frac{4}{3} - 4 = \underline{-\frac{16}{3}}$

$\underline{\underline{x_t = -\frac{16}{3} 2^t + \frac{4}{3} 2^{-t} + 4 \cdot 3^t}}$

b) $-\alpha x_{t+1} + (1+\alpha^2) x_t - \alpha x_{t-1} = K \beta^t \qquad \alpha, \beta > 0, \quad \alpha \neq 1, \beta \neq \alpha, \beta \neq \frac{1}{\alpha}$

$x_{t+1} - \frac{1+\alpha^2}{\alpha} x_t + x_{t-1} = -\frac{K}{\alpha} \beta^t = -\frac{K}{\alpha} \beta \beta^{t-1}$

$\Rightarrow \boxed{x_{t+2} - \frac{1+\alpha^2}{\alpha} x_{t+1} + x_t = -\frac{\beta}{\alpha} K \beta^t} \qquad \frac{(1+\alpha^2)^2}{\alpha^2} - 4 = \frac{1}{\alpha^2} + 2 + \alpha^2 - 4 = \alpha^2 + \frac{1}{\alpha^2} - 2$

$\circ > 0$ if $\alpha^2 + \frac{1}{\alpha^2} - 2 > 0 \quad \alpha^4 - 2\alpha^2 + 1 > 0 \quad a^2 - 2a + 1 > 0$ where $a = \alpha^2$

i.e. $(a-1)^2 > 0$ will always happen since $\alpha \neq 1$.

Thus we are in case (I) $\Rightarrow x_t = A m_1^t + B m_2^t, \; m_{1,2} = \frac{1+\alpha^2}{2\alpha} \pm \frac{1}{2}\sqrt{\alpha^2 + \frac{1}{\alpha^2} - 2} = \frac{1+\alpha^2}{2\alpha} \pm \frac{1}{2}\sqrt{\frac{(\alpha^2-1)^2}{\alpha^2}}$

$m_{1,2} = \frac{1+\alpha^2}{2\alpha} \pm \frac{(\alpha^2-1)}{2\alpha}$ $\quad \underline{m_1 = \alpha} \quad \underline{m_2 = \alpha^{-1}} \quad \Rightarrow \underline{x_t = A\alpha^t + B\alpha^{-t}}$

$y_t = C\beta^t$

$$C\beta^{t+2} - \frac{1+\alpha^2}{\alpha} C\beta^{t+1} + C\beta^t = -\frac{\beta}{\alpha} K \beta^t$$

$$C\left[\beta^2 - \frac{1+\alpha^2}{\alpha}\beta + 1\right] = -\frac{\beta}{\alpha} K \qquad \underline{C = \frac{-\beta K}{\alpha\beta^2 - (1+\alpha^2)\beta + \alpha}}$$

$$\Rightarrow \underline{\underline{x_t = A\alpha^t + B\alpha^{-t} - \frac{K}{\alpha\beta^2 - (1+\alpha^2)\beta + \alpha}\beta^{t+1}}}$$

$$\boxed{11-01}$$

$$\max\left\{ \sum_{t=0}^{T-1} (-u_t^2) - x_T^2 \right\} \quad \text{s.t.} \quad x_{t+1} = x_t + u_t \,, \quad u_t \in \mathbb{R}$$

a) $J_T(x) = \max\{-x_T^2\} = \underline{-x_T^2}$, $\underline{u_T \text{ any } v \in \mathbb{R}}$

$J_{T-1}(x) = \max\{-v^2 - (x+v)^2\} \Rightarrow -2v - 2(x+v) = 0 \quad \underline{v = -\frac{1}{2}x} \quad \underline{u_{T-1} = -\frac{1}{2}x}$

$J_{T-1}(x) = -\frac{1}{4}x^2 - (\frac{1}{4}x^2) \quad \underline{J_{T-1}(x) = -\frac{1}{2}x^2}$

$\cancel{J_{T-2}(x) = -v^2 - \frac{1}{2}(x+v)^2} \quad J_{T-2}(x) = \max\limits_v \{-v^2 - \frac{1}{2}(x+v)^2\}$

$\Rightarrow -2v - (x+v) = 0 \quad v = -\frac{1}{3}x \quad \underline{u_{T-2} = -\frac{1}{3}x} \quad J_{T-2} = -\frac{1}{9}x^2 - \frac{1}{2}\frac{4}{9}x^2 \quad \underline{J_{T-2}(x) = -\frac{1}{3}x^2}$

b) Guess $J_{T-k}(x) = -\frac{1}{k+1}x^2$, $u_{T-k} = -\frac{1}{k+1}x$

$J_{T-k-1}(x) = \max\limits_v \{-v^2 - \frac{1}{k+1}(x+v)^2\} \Rightarrow -2v - \frac{2}{k+1}(x+v) = 0$

$v + \frac{v}{k+1} = -\frac{x}{k+1} \quad v = \frac{-\frac{x}{k+1}}{(1+\frac{1}{k+1})} = -\frac{\frac{x}{k+1}}{\frac{k+2}{k+1}} \quad \underline{v = -\frac{x}{k+2}}$

$J_{T-k-1}(x) = -\frac{x^2}{(k+2)^2} - \frac{1}{k+1}\left(x - \frac{x}{k+2}\right)^2 = -\frac{x^2}{(k+2)^2} - \frac{x^2(k+1)}{(k+2)^2} = \underline{-\frac{x^2}{k+2}}$

This gives $u_{T-(k+1)} = -\frac{x}{(k+1)+1} \quad J_{T-(k+1)}(x) = -\frac{x^2}{(k+1)+1}$

$\Rightarrow$ By induction $\underline{J_{T-k}(x) = -\frac{1}{k+1}x^2} \quad \underline{u_{T-k} = -\frac{1}{k+1}x}$

$\boxed{11\text{-}03}$

$\max \sum_{t=0}^{T-1} \ln u_t + \ln x_T$ s.t. $x_{t+1} = x_t - u_t$, $x_0 > 0$, $u_t \in (0, x_t)$

a) $J_T(x) = \ln x$, u_T any $u \in (0,x)$

$J_{T-1}(x) = \max_{u \in (0,x)} \{ \ln u + \ln(x-u) \} \Rightarrow \frac{1}{u} = \frac{1}{x-u}$ $x-u = u$ $u = \frac{1}{2}x$ $u_{T-1} = \frac{1}{2}x$

$J_{T-1}(x) = \ln\left(\frac{1}{2}x\right) + \ln\left(\frac{1}{2}x\right)$ $J_{T-1}(x) = 2\ln\left(\frac{1}{2}x\right)$

$J_{T-2}(x) = \max_{u \in (0,x)} \{ \ln u + 2\ln\left(\frac{1}{2}(x-u)\right) \} \Rightarrow \frac{1}{u} = \frac{2}{\frac{1}{2}(x-u)} \cdot \frac{1}{2} = \frac{2}{x-u}$

$x - u = 2u$ $u = \frac{1}{3}x$ $u_{T-2} = \frac{1}{3}x$

$J_{T-2}(x) = \ln\left(\frac{1}{3}x\right) + 2\ln\left(\frac{1}{3}x\right)$ $J_{T-2}(x) = 3\ln\left(\frac{1}{3}x\right)$

Guess ~~$J_T(x)$~~ $J_{T-k}(x) = (k+1)\ln\left(\frac{x}{k+1}\right)$, $u_{T-k} = \frac{x}{k+1}$

$\Rightarrow$ $J_{T-(k+1)}(x) = \max_{u \in (0,x)} \left\{ \ln u + (k+1)\ln\left(\frac{x-u}{k+1}\right) \right\} \Rightarrow \frac{1}{u} = (k+1)\frac{1}{\frac{x-u}{(k+1)}} \cdot \frac{1}{k+1} = \frac{k+1}{x-u}$

$\Rightarrow$ $x - u = (k+1)u$ $u = \frac{x}{(k+2)}$ $u_{T-(k+1)} = \frac{x}{(k+1)+1}$

$J_{T-(k+1)}(x) = \ln \frac{x}{(k+1)+1} + (k+1)\ln\left(\frac{x - \frac{x}{(k+1)+1}}{k+1}\right) = ((k+1)+1)\ln\left(\frac{x}{(k+1)+1}\right)$

$\Rightarrow$ Thus by induction $u_{T-k} = \frac{x}{k+1}$ $J_{T-k} = (k+1)\ln\left(\frac{x}{k+1}\right)$

$\boxed{11\text{-}04}$

$\max \sum_{t=0}^{T} x_t^2(1+u_t)$ s.t. $x_{t+1} = x_t(1-u_t)$, $u_t \in [0,1]$, x_0 given

a) $J_T(x) = \max_{v\in[0,1]} \{x^2(1+v)\} \Rightarrow \underline{\underline{u_T = 1}}$, $\underline{\underline{J_T(x) = 2x^2}}$ (If $x=0$, v is fixed any $v\in[0,1]$)

$J_{T-1}(x) = \max_{v\in[0,1]} \{x^2(1+v) + 2x^2(1-v)^2\}$

Danger! This function is convex! Do not use F.O.C.!

Thus, choose a corner solution, either $v=0$ or $v=1$

$v=0$ gives $J_{T-1}(x) = x^2 + 2x^2 = 3x^2$

$v=1$ gives $J_{T-1}(x) = 2x^2$ in general $3x^2 \geq 2x^2$

$\Rightarrow$ $v=0$ optimal $\underline{\underline{u_{T-1} = 0}}$ (If $x=0$, any $v\in[0,1]$ optimal)

$\underline{\underline{J_{T-1}(x) = 3x^2}}$

b) Assume $J_{T-n}(x) = (n+2)x^2$

$J_{T-(n+1)}(x) = \max_{v\in[0,1]} \{x^2(1+v) + (n+2)x^2(1-v)^2\}$

Corner solution as before $v=0$ gives $J_{T-(n+1)}(x) = (n+3)x^2$

$v=1$ gives $J_{T-(n+1)}(x) = 2x^2$

$\Rightarrow$ $v=0$ with $J_{T-(n+1)}(x) = (n+3)x^2$ optimal

$\Rightarrow$ $\underline{\underline{J_{T-n}(x) = (n+2)x^2}}$ $\underline{\underline{u_T = 1,\ u_{T-n} = 0 \text{ for all } n>0}}$

$J_0(x) = J_{T-T}(x) = (T+2)x^2 = (T+2)x_0^2 \Rightarrow \underline{\underline{J_0(x) = (T+2)x_0^2}}$

~~$x_t = x_{t-1}$ for all~~ $\underline{\underline{x_t = x_0 \text{ for all } t\in[0,T]}}$