11-01

$$\max\left\{\sum_{t=0}^{T-1}(-u_t^2) - x_T^2\right\} \quad \text{s.t.} \quad x_{t+1} = x_t + u_t \quad , u_t \in \mathbb{R}$$

a) $J_T(x) = \max\{-x^2\} = -x^2$, u_T any $v \in \mathbb{R}$

$J_{T-1}(x) = \max\{-v^2 - (x+v)^2\} \Rightarrow -2v - 2(x+v) = 0 \quad v = -\frac{1}{2}x \quad u_{T-1} = -\frac{1}{2}x$

$J_{T-1}(x) = -\frac{1}{4}x^2 - (\frac{1}{4}x^2) \quad J_{T-1}(x) = -\frac{1}{2}x^2$

~~$J_{T-2}(x) = -v^2 - \frac{1}{2}(x+v)^2$~~ $J_{T-2}(x) = \max_v\{-v^2 - \frac{1}{2}(x+v)^2\}$

$\Rightarrow -2v - (x+v) = 0 \quad v = -\frac{1}{3}x \quad u_{T-2} = -\frac{1}{3}x \quad J_{T-2} = -\frac{1}{9}x^2 - \frac{1}{2}\frac{4}{9}x^2 \quad J_{T-2}(x) = -\frac{1}{3}x^2$

b) Guess $J_{T-k}(x) = -\frac{1}{k+1}x^2$, $u_{T-k} = -\frac{1}{k+1}x$

$J_{T-k-1}(x) = \max_v\{-v^2 - \frac{1}{k+1}(x+v)^2\} \Rightarrow -2v - \frac{2}{k+1}(x+v) = 0$

$v + \frac{v}{k+1} = -\frac{x}{k+1} \quad v = \frac{-\frac{x}{k+1}}{(1+\frac{1}{k+1})} = -\frac{\frac{x}{k+1}}{\frac{k+2}{k+1}} \quad v = -\frac{x}{k+2}$

$J_{T-k-1}(x) = -\frac{x^2}{(k+2)^2} - \frac{1}{k+1}\left(x - \frac{x}{k+2}\right)^2 = -\frac{x^2}{(k+2)^2} - \frac{x^2(k+1)}{(k+2)^2} = -\frac{x^2}{k+2}$

This gives $u_{T-(k+1)} = -\frac{x}{(k+1)+1} \quad J_{T-(k+1)}(x) = -\frac{x^2}{(k+1)+1}$

$\Rightarrow$ By induction $J_{T-k}(x) = -\frac{1}{k+1}x^2 \quad u_{T-k} = -\frac{1}{k+1}x$

6

11-03

$$\max \sum_{t=0}^{T-1} \ln u_t + \ln x_T \quad \text{s.t.} \quad x_{t+1} = x_t - u_t \ , \ x_0 > 0 \ , \ u_t \in (0, x_t)$$

a) $J_T(x) = \ln x$, u_T any $u \in (0, x)$

$J_{T-1}(x) = \max_{u \in (0,x)} \{ \ln u + \ln(x-u) \} \Rightarrow \frac{1}{u} = \frac{1}{x-u} \quad x-u = u \quad u = \frac{1}{2}x \quad u_{T-1} = \frac{1}{2}x$

$J_{T-1}(x) = \ln\left(\frac{1}{2}x\right) + \ln\left(\frac{1}{2}x\right) \quad J_{T-1}(x) = 2\ln\left(\frac{1}{2}x\right)$

$J_{T-2}(x) = \max_{u \in (0,x)} \{ \ln u + 2\ln\left(\frac{1}{2}(x-u)\right) \} \Rightarrow \frac{1}{u} = \frac{2}{\frac{1}{2}(x-u)} \cdot \frac{1}{2} = \frac{2}{x-u}$

$x - u = 2u \quad u = \frac{1}{3}x \quad u_{T-2} = \frac{1}{3}x$

$J_{T-2}(x) = \ln\left(\frac{1}{3}x\right) + 2\ln\left(\frac{1}{3}x\right) \quad J_{T-2}(x) = 3\ln\left(\frac{1}{3}x\right)$

Guess $J_{T-k}(x) = (k+1)\ln\left(\frac{x}{k+1}\right)$, $u_{T-k} = \frac{x}{k+1}$

$\Rightarrow J_{T-(k+1)}(x) = \max_{u \in (0,x)} \{ \ln u + (k+1)\ln\left(\frac{x-u}{k+1}\right) \} \Rightarrow \frac{1}{u} = (k+1)\frac{1}{\frac{x-u}{k+1}} \frac{1}{k+1} = \frac{k+1}{x-u}$

$\Rightarrow x - u = (k+1)u \quad u = \frac{x}{(k+2)} \quad u_{T-(k+1)} = \frac{x}{(k+1)+1}$

$J_{T-(k+1)}(x) = \ln\frac{x}{(k+1)+1} + (k+1)\ln\left(\frac{x - \frac{x}{(k+1)+1}}{k+1}\right) = ((k+1)+1)\ln\left(\frac{x}{(k+1)+1}\right)$

$\Rightarrow$ Thus by induction $u_{T-k} = \frac{x}{k+1}$ $\quad J_{T-k} = (k+1)\ln\left(\frac{x}{k+1}\right)$

11-04

$\max \sum_{t=0}^{T} x_t^2(1+u_t)$ s.t. $x_{t+1} = x_t(1-u_t)$, $u_t \in [0,1]$, x_0 given

a) $J_T(x) = \max_{v\in[0,1]} \{x^2(1+v)\} \Rightarrow u_T = 1$, $J_T(x) = 2x^2$ (If $x=0$, v in fact any $v \in [0,1]$)

$J_{T-1}(x) = \max_{v\in[0,1]} \{x^2(1+v) + 2x^2(1-v)^2\}$

Danger! This function is convex! Do not use F.O.C.!

Thus, choose a corner solution, either $v=0$ or $v=1$

$v=0$ gives $J_{T-1}(x) = x^2 + 2x^2 = 3x^2$

$v=1$ gives $J_{T-1}(x) = 2x^2$ in general $3x^2 \geq 2x^2$

$\Rightarrow$ $v=0$ optimal $u_{T-1} = 0$ (If $x=0$, any $v \in [0,1]$ optimal)

$J_{T-1}(x) = 3x^2$

b) Assume $J_{T-n}(x) = (n+2)x^2$

$J_{T-(n+1)}(x) = \max_{v\in[0,1]} \{x^2(1+v) + (n+2)x^2(1-v)^2\}$

Corner solution as before $v=0$ gives $J_{T-(n+1)}(x) = (n+3)x^2$

$v=1$ gives $J_{T-(n+1)}(x) = 2x^2$

$\Rightarrow$ $v=0$ with $J_{T-(n+1)}(x) = (n+3)x^2$ optimal

$\Rightarrow$ $J_{T-n}(x) = (n+2)x^2$ $u_T = 1$, $u_{T-n} = 0$ for all $n > 0$

$J_0(x) = J_{T-T}(x) = (T+2)x^2 = (T+2)x_0^2 \Rightarrow J_0(x) = (T+2)x_0^2$

~~$x_t = x_{t-1}$ for all~~ $x_t = x_0$ for all $t \in [0,T]$

Seminar 10

Problem by Framstad

$$J_0(x) = \max_{u_t \in [0,1]} \left\{ \sum_{t=0}^{T-1} 2q^t \sqrt{u_t x_t} + pq^T \sqrt{x_T} \right\}, \quad x_{t+1} = x_t(1-u_t) \quad x_0 = x \geq 0$$

$q \in (0,1)$, $p \in \mathbb{R}$

a) $J_T(x) = \max_{v \in [0,1]} \{ pq^T \sqrt{x} \} = pq^T \sqrt{x}$ $\quad$ u_T any number in $[0,1]$

$$J_{T-1}(x) = \max_{v \in [0,1]} \{ 2q^{T-1} \sqrt{ux} + pq^T \sqrt{x(1-u)} \} = q^{T-1} \max_{v \in [0,1]} \{ 2\sqrt{ux} + pq\sqrt{x(1-u)} \}$$

$$\Rightarrow (ux)^{-\frac{1}{2}} x + \frac{1}{2} pq (x(1-u))^{-\frac{1}{2}} (-x) = 0 \quad \text{(Indifferent if } x=0)$$

$x > 0 \Rightarrow (1-v)^{\frac{1}{2}} = \frac{pq}{2} u^{\frac{1}{2}}$

$$1-u = \frac{p^2q^2}{4} u \qquad u = \frac{1}{1+\frac{p^2q^2}{4}} \qquad u_{T-1} = \frac{4}{p^2q^2+4} \qquad 1-u_{T-1} = \frac{p^2q^2}{p^2q^2+4}$$

$$\Rightarrow J_{T-1}(x) = \sqrt{x}\, q^{T-1} \left(2\frac{2}{\sqrt{p^2q^2+4}} + pq \frac{pq}{\sqrt{p^2q^2+4}} \right)$$

$$J_{T-1}(x) = \sqrt{4+p^2q^2}\; q^{T-1} \sqrt{x}$$

Assume $J_{T-k}(x) = a_{T-k} q^{T-k} \sqrt{x}$

$$J_{T-(k+1)}(x) = q^{T-(k+1)} \max_{v \in [0,1]} \{ 2\sqrt{xv} + a_{T-k} q \sqrt{x(1-v)} \}$$

$$\Rightarrow v^{-\frac{1}{2}} + \frac{1}{2} a_{T-k} q (1-v)^{-\frac{1}{2}} (-1) = 0 \qquad (1-v)^{\frac{1}{2}} = \frac{1}{2} a_{T-k} q\, v^{\frac{1}{2}}$$

$$(1-v) = \frac{1}{4} a_{T-k}^2 q^2 v \qquad u_{T-(k+1)} = \frac{1}{1+\frac{1}{4}a_{T-k}^2 q^2} = \frac{4}{4+a_{T-k}^2 q^2} \qquad 1-u = \frac{a_{T-k}^2 q^2}{4+a_{T-k}^2 q^2}$$

$$\Rightarrow J_{T-(k+1)}(x) = q^{T-(k+1)} \sqrt{x} \left(\frac{2 \cdot 2}{\sqrt{4+a_{T-k}^2 q^2}} + a_{T-k} q \frac{a_{T-k} q}{\sqrt{4+a_{T-k}^2 q^2}} \right) = \sqrt{4+a_{T-k}^2 q^2}\; q^{T-(k+1)} \sqrt{x}$$

$J_{T-(k+1)}(x) = a_{T-(k+1)} q^{T-(k+1)} \sqrt{x}$ $\quad \Rightarrow J_{T-k}(x) = a_{T-k} q^{T-k} \sqrt{x}$ by induction

$$\Rightarrow J_t(x) = a_t q^t \sqrt{x}$$

b)

~~$J(x) = \max_{u\in[0,1]} \{ 2q\sqrt{ux} +$~~

$$J(x) = \max_{u\in[0,1]} \{ 2\sqrt{ux} + q\, J(x(1-u)) \}$$ Bellman equation

↳ notice, q is the discounting factor of this problem

Ⓐ $J(x) = A\sqrt{x}$

$\Rightarrow$ Ⓑ $\max_{u\in[0,1]} \{ 2\sqrt{ux} + q A\sqrt{x(1-u)} \}$

F.o.c. a $u^{-\frac{1}{2}} + \frac{1}{2} q A (1-u)^{-\frac{1}{2}} (-1) = 0$

$\Rightarrow u = 4 q^{-2} A^{-2} (1-u)$ $\quad \frac{1}{4} q^2 A^2 u = 1-u$ $\quad u = \frac{1}{1+\frac{1}{4}q^2A^2}$ $\quad u = \frac{4}{4+q^2A^2}$

$\Rightarrow$ Ⓑ $\sqrt{x}\left[2\frac{4}{\sqrt{4+q^2A^2}} + qA \frac{qA}{\sqrt{4+q^2A^2}} \right] = \sqrt{4+q^2A^2}\,\sqrt{x}$

$\Rightarrow A\sqrt{x} = \sqrt{4+q^2A^2}\,\sqrt{x} \quad \Rightarrow \quad A^2 = 4 + q^2A^2 \quad A^2[1-q^2] = 4$

$A = \pm \frac{2}{\sqrt{1-q^2}} \quad A > 0 \Rightarrow$ $$A = \frac{2}{\sqrt{1-q^2}}$$

$\Rightarrow J(x) = \frac{2}{\sqrt{1-q^2}}\sqrt{x}$ satisfies the Bellman equation

8-01

$\max \int_0^1 (2x e^{-t} - 2x\dot{x} - \dot{x}^2)\,dt \quad x(0)=0 \quad x(1)=1$

a) $\frac{\partial F}{\partial x} = 2e^{-t} - 2\dot{x} \qquad \frac{\partial F}{\partial \dot{x}} = -2x - 2\dot{x}$

$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\dot{x} - 2\ddot{x}$

$\Rightarrow$ Euler equation $\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)$ gives $2e^{-t} - 2\dot{x} + 2\dot{x} + 2\ddot{x} = 0$

$\Rightarrow \boxed{\ddot{x} = -e^{-t}}$

b) $\dot{x} = e^{-t} + C \qquad x = -e^{-t} + Ct + D$

$x(0) = -1 + D = 0 \Rightarrow D = 1$

$x(1) = -e^{-1} + C + 1 = 1 \Rightarrow C = e^{-1} \Rightarrow x(t) = -e^{-t} + e^{-1}t + 1$

8-02

$\max \int_0^1 (4xe^{-t} - 5x^2 - \dot{x}^2)e^{-4t}\,dt \qquad x(0) = \frac{5}{3} \qquad x(1) = 2e^{-1}$

$\frac{\partial F}{\partial x} = (4e^{-t} - 10x)e^{-4t} \qquad \frac{\partial F}{\partial \dot{x}} = -2\dot{x}e^{-4t} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\ddot{x}e^{-4t} - 2\dot{x}e^{-4t}(-4)$

$= -2e^{-4t}(\ddot{x} - 4\dot{x})$

Euler eq.: $4e^{-t} - 10x + 2\ddot{x} - 8\dot{x} = 0 \qquad \boxed{\ddot{x} - 4\dot{x} - 5x = -2e^{-t}}$

$\frac{1}{4}4^2 + 5 = 9 > 0 \Rightarrow x(t) = Ae^{r_1 t} + Be^{r_2 t} \qquad r_{1,2} = +2 \pm \sqrt{9} \qquad r_1 = 5 \quad r_2 = -1$

$x(t) = Ae^{5t} + Be^{-t} \qquad u^*(t) = \frac{-2}{1+4-5}e^{-t} = \frac{1}{4}e^{-t}$

$\Rightarrow x^*(t) = Ae^{t} + Be^{-5t} + \frac{1}{4}e^{-t} \qquad x^*(0) = A + B + \frac{1}{4} = \frac{5}{3} \qquad A = \frac{17}{20} - B$

$x^*(1) = \left(\frac{17}{20} - B\right)e + Be^{-5} + \frac{1}{4}e^{-1} = 2e^{-1} \qquad \boxed{B = \frac{\frac{7}{4}e^{-1} - \frac{17}{20}e}{e^{-5} - e}} \qquad \boxed{A = \frac{17}{20} - \frac{\frac{7}{4}e^{-1} - \frac{17}{20}e}{e^{-5} - e}}$

18-02

$$\max \int_0^1 (4x e^{-t} - 5x^2 - \dot{x}^2) e^{-4t} \, dt \qquad x(0) = \frac{5}{3} \quad x(1) = 2e^{-1}$$

$$\frac{\partial F}{\partial x} = (4e^{-t} - 10x) e^{-4t} \qquad \frac{\partial F}{\partial \dot{x}} = -2\dot{x} e^{-4t}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\ddot{x} e^{-4t} + 8\dot{x} e^{-4t} = 2e^{-4t}(4\dot{x} - \ddot{x})$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = e^{-4t}\left[4e^{-t} - 10x - 2(4\dot{x} - \ddot{x})\right] = 0$$

$$\boxed{\ddot{x} - 4\dot{x} - 5x = -2e^{-t}}$$ Euler equation

$$\frac{1}{4}(-4)^2 - 5 = 4 + 5 = 9 > 0$$

$\Rightarrow$ Homogeneous solution: $x(t) = Ae^{r_1 t} + Be^{r_2 t}$ $\quad r_{1,2} = 2 \pm \sqrt{9}$

$$x(t) = Ae^{5t} + Be^{-t}$$

Nonhomogeneous: $f(t) = -2e^{-t}$, i.e. of the form pe^{qt} where $p = -2$, $q = -1$.

Notice $q = -1$ is single root of $q^2 - 4q - 5$

$\Rightarrow$ $u^* = Cte^{-t}$. Must find C, insert in Euler equation.

$$\dot{u}^* = Ce^{-t}[1-t] \qquad \ddot{u}^* = Ce^{-t}[t-2]$$

Thus $\quad Ce^{-t}[t-2] - 4Ce^{-t}[1-t] - 5Cte^{-t} = -2e^{-t}$

$$\Rightarrow Ce^{-t}[t - 2 - 4 + 4t - 5t] = Ce^{-t}[-6] = -2e^{-t} \quad \Rightarrow \boxed{C = \frac{1}{3}}$$

Thus $\quad x(t) = Ae^{5t} + Be^{-t} + \frac{1}{3}te^{-t}$

$x(0) = A + B = \frac{5}{3} \qquad B = \frac{5}{3} - A \qquad x(1) = Ae^{5} + \left(\frac{5}{3} - A\right)e^{-1} + \frac{1}{3}e^{-1} = 2e^{-1}$

$\Rightarrow A(e^{5} - e^{-1}) = 0 \qquad \underline{A = 0} \qquad \underline{B = \frac{5}{3}}$

$$\underline{\underline{x(t) = \frac{1}{3}e^{-t}[5 + t]}}$$

8-03

$$\max \int_0^T \left(\frac{1}{100}tx - \dot{x}^2\right) e^{-\frac{t}{10}}\,dt \quad , x(0)=0 \quad , x(T)=S$$

a) $\frac{\partial F}{\partial x} = \frac{1}{100} t e^{-\frac{t}{10}}$ $\quad \frac{\partial F}{\partial \dot{x}} = -2\dot{x} e^{-\frac{t}{10}}$ $\quad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\left[\ddot{x} e^{-\frac{t}{10}} + \dot{x} e^{-\frac{t}{10}}\left(-\frac{1}{10}\right)\right]$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \boxed{e^{-\frac{t}{10}}\left[\frac{t}{100} + 2\left(\ddot{x} - \frac{\dot{x}}{10}\right)\right] = 0}$$ Euler equation

$\Rightarrow \ddot{x} - \frac{1}{10}\dot{x} = -\frac{t}{200}$ Taking integral gives $\dot{x} - \frac{1}{10}x = -\frac{t^2}{400} + C$

$$\frac{d}{dt}\left(x e^{-\frac{1}{10}t}\right) = -\frac{t^2}{400} e^{-\frac{1}{10}t} + C e^{-\frac{1}{10}t} \quad \Rightarrow x e^{-\frac{1}{10}t} = -\frac{1}{400}\int t^2 e^{-\frac{1}{10}t}dt + C\int e^{-\frac{1}{10}t}dt + D \quad (*)$$

$$\int t^2 e^{-\frac{1}{10}t}dt = -10t^2 e^{-\frac{1}{10}t} + \int 20t e^{-\frac{1}{10}t}dt = -10t^2 e^{-\frac{1}{10}t} + 20\left[-10t e^{-\frac{1}{10}t} + \int 10 e^{-\frac{1}{10}t}dt\right]$$

$$= -10 e^{-\frac{1}{10}t}\left[t^2 + 20t + 200\right]$$

$$\int e^{-\frac{1}{10}t}dt = -10 e^{-\frac{1}{10}t}$$

$\Rightarrow$ (*) gives $x(t) = \frac{1}{40}\left[t^2 + 20t + 200\right] - 10C + D e^{\frac{1}{10}t}$

$\Rightarrow x(t) = \frac{1}{40}t^2 + \frac{1}{2}t + A + D e^{\frac{1}{10}t}$ $\quad (A = 5 - 10C)$

b) $T=10$, $S=20$ $\quad x(0) = A + D = 0 \quad A = -D$

$x(10) = \frac{5}{2} + 5 - D + De = 20 \quad D = \frac{15 - \frac{5}{2}}{e-1}$ $\quad \boxed{D = \frac{25}{2(e-1)}}$ $\boxed{A = -\frac{25}{2(e-1)}}$

$$x(t) = \frac{1}{40}t^2 + \frac{1}{2}t - \frac{25}{2(e-1)} + \frac{25}{2(e-1)} e^{\frac{1}{10}t}$$

$\frac{1}{100}tx - \dot{x}^2$ is concave w.r.t. $(x, \dot{x})$ since $\frac{\partial^2 F}{\partial x^2} = 0$, $\frac{\partial^2 F}{\partial \dot{x}^2} = -2 < 0$ $\quad \frac{\partial^2 F}{\partial x \partial \dot{x}} = 0$

$\Rightarrow \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial \dot{x}^2} - \left(\frac{\partial^2 F}{\partial x \partial \dot{x}}\right)^2 = 0$, thus solution is optimal.

(8-06)

$$\min \int_0^T p x^2 + q \frac{1}{b^2} (\dot{x} - a x)^2 \, dt \qquad x(0) = x_0 \quad x(T) = x_T$$

a) $\frac{\partial F}{\partial x} = 2px - 2q \frac{a}{b^2} (\dot{x} - ax)$ $\qquad \frac{\partial F}{\partial \dot{x}} = 2q \frac{1}{b^2} (\dot{x} - ax)$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2q \frac{1}{b^2} (\ddot{x} - a\dot{x})$$

Euler equation: $2px - 2q\frac{a}{b^2} (\dot{x} - ax) - 2q\frac{1}{b^2} (\ddot{x} - a\dot{x}) = 0$

$$\frac{q}{b^2} \ddot{x} - px - \frac{q a^2}{b} x = 0$$

$$\boxed{\ddot{x} - x\left(\frac{p}{q} b^2 + \frac{a^2}{b}\right) = 0}$$

$\Rightarrow x(t) = A e^{r_1 t} + B e^{r_2 t}$ $\quad r_{1,2} = \pm\sqrt{\frac{p}{q} b^2 + \frac{a^2}{b}}$ if $\frac{p}{q} b^2 + \frac{a^2}{b} > 0$, $x(t) = (A + Bt)$ if $\frac{p}{q} b^2 + \frac{a^2}{b} = 0$

b) ~~p = 0 ... = 1~~ $p = 0 \quad q = a = b = T = 1 \quad x_0 = 0 \quad x_T = 1 \quad \Rightarrow \frac{p}{q} b^2 + \frac{a^2}{b} > 0$

$r_{1,2} = \pm\sqrt{1} = \pm 1 \quad \Rightarrow x(t) = A e^{t} + B e^{-t}$

$x(0) = A + B = 0 \qquad A = -B$

$x(1) = B(-e + e^{-1}) = 1 \qquad B = \frac{1}{e^{-1} - e} \qquad A = \frac{1}{e - e^{-1}}$

$x(t) = \frac{1}{e - e^{-1}} \left[e^{t} - e^{-t} \right] \qquad x(t) = \frac{e^{t} - e^{-t}}{e - e^{-1}}$

Problem is $\min \int_0^1 (\dot{x} - x)^2 \, dt \Rightarrow \frac{\partial^2 F}{\partial x^2} = 2 \quad \frac{\partial^2 F}{\partial \dot{x}^2} = 2 \quad \frac{\partial^2 F}{\partial x \partial \dot{x}} = -2 \Rightarrow$ Problem convex

ok since min