Seminar 3 - ECON 4140

Problem 2, exam 05

$\max \{xy\}$ s.t. $\begin{cases} x^2+ry^2 \le m \\ x \ge 1 \end{cases}$, $r>0, m>1$.

a) $\mathcal{L}(x,y) = xy - \lambda(x^2+ry^2-m) - \mu(1-x)$

$$\boxed{\begin{array}{ll} (1) & \frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda x + \mu = 0 \\ (2) & \frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda r y = 0 \\ (3) & \lambda \ge 0 \ (\lambda = 0 \text{ if } x^2+ry^2<m) \\ (4) & \mu \ge 0 \ (\mu = 0 \text{ if } x>1) \\ (5) & x^2+ry^2 \le m \\ (6) & x \ge 1 \end{array}}$$

b) • Assume $x^2+ry^2=m$ and $\boxed{x=1}$

$\Rightarrow ry^2 = m-1$ $\quad y^2 = \frac{m-1}{r}$ $\quad y = \pm\sqrt{\frac{m-1}{r}}$

(2) gives $\lambda = \frac{1}{2ry}$, thus $\boxed{y = \sqrt{\frac{m-1}{r}}}$ must hold since $\lambda \ge 0$.

• (1) gives $\mu = -\sqrt{\frac{m-1}{r}} + \frac{1}{\sqrt{(m-1)r}} \ge 0$ iff $\frac{1}{\sqrt{(m-1)r}} \ge \frac{\sqrt{m-1}}{\sqrt{r}}$ i.e. $m-1 \le 1$ i.e. $m \le 2$

$\boxed{\text{Thus, when } m \le 2,\ (x,y) = \left(1, \sqrt{\tfrac{m-1}{r}}\right) \text{ is candidate}}$

• Assume $x^2+ry^2=m$, $x>1$ $\Rightarrow \mu = 0$

(1) gives $y = 2\lambda x$, (2) gives $x - 2\lambda r(2\lambda x) = 0$

$\Rightarrow x(1-4\lambda^2 r) = 0 \Rightarrow \lambda^2 = \frac{1}{4r}$ $\quad \lambda = \frac{1}{2\sqrt{r}}$

$\Rightarrow y = \frac{x}{\sqrt{r}} \Rightarrow$ (5) gives $x^2+x^2=m$, i.e. $2x^2=m$ $\quad x = \pm\sqrt{\frac{m}{2}}$

Since $x>1$, must have $x = \sqrt{\frac{m}{2}}$, which is ok iff $m>2$.

$\boxed{\text{Thus, when } m>2,\ (x,y) = \left(\sqrt{\tfrac{m}{2}}, \sqrt{\tfrac{m}{2r}}\right) \text{ is a candidate}}$

• Assume $x^2+ry^2<m$ and $\boxed{x=1}$ $\Rightarrow$ $\lambda=0$

This case is impossible since by (2) we get $1=0$.

• Assume $x^2+ry^2<m$ and $x>1$ $\Rightarrow$ $\lambda=0$ and $\mu=0$

(2) gives $x=0$, which violates $x>1$ $\Rightarrow$ impossible.

Not that for each value $m>1$ we in fact only have one candidate, since the two possible candidates we get applies for non-overlapping values of m.

Finally we see that $\mathcal{L}(x,y)$ is sum of concave functions, and is thus concave, which means the solution to the problem is

$$(x,y)=\begin{cases}\left(1,\sqrt{\frac{m-1}{r}}\right) & \text{for } m\le 2\\ \left(\sqrt{\frac{m}{2}},\sqrt{\frac{m}{2r}}\right) & \text{for } m>2\end{cases}$$

$\boxed{\text{Problem 3 exam 07}}$

$$\max \; -(x-6)^2-(y-5)^2 \quad \text{s.t.} \begin{cases} x^2+y^2 \le 25 \\ a(x-3)+y \le 4 \end{cases} \qquad a \neq \frac{3}{4}$$

a) $\mathcal{L} = -(x-6)^2-(y-5)^2 - \lambda(x^2+y^2-25) - \mu(a(x-3)+y-4)$

$$\boxed{\begin{array}{ll} (1) & \frac{\partial \mathcal{L}}{\partial x} = -2(x-6) - 2\lambda x - \mu a = 0 \\ (2) & \frac{\partial \mathcal{L}}{\partial y} = -2(y-5) - 2\lambda y - \mu = 0 \\ (3) & \lambda \ge 0 \quad (\lambda = 0 \text{ if } x^2+y^2 < 25) \\ (4) & \mu \ge 0 \quad (\mu = 0 \text{ if } a(x-3)+y < 4) \\ (5) & x^2+y^2 \le 25 \\ (6) & a(x-3)+y \le 4 \end{array}}$$

b) ~~when~~ For what a will $(x,y)=(3,4)$ be optimal?

If $x=3$ and $y=4$, then (5) gives $x^2+y^2 = 9+16 = 25$, that is (5) is binding.

And (6) gives $a \cdot 0 + 4 = 4$, that is (6) is binding.

$\Rightarrow$ No more information about λ and μ yet.

(1) gives $\quad 6 - 6\lambda - \mu a = 0 \quad \Rightarrow \quad \boxed{\mu = \frac{6(1-\lambda)}{a}}$

(2) then gives $\quad 2 - 8\lambda = \frac{6(1-\lambda)}{a} \qquad 2a - 8a\lambda = 6 - 6\lambda \qquad 2(a-3) = 2\lambda(4a-3)$

that is, $\boxed{\lambda = \frac{a-3}{4a-3}} \Rightarrow \mu = \frac{6\left(\frac{4a-3}{4a-3} - \frac{a-3}{4a-3}\right)}{a} = \frac{6\left(\frac{3a}{4a-3}\right)}{a} = \frac{18}{4a-3}$

$\boxed{\mu = \frac{18}{4a-3}}$ Need $\mu \ge 0$ which is ok when $4a-3 > 0$ or $a > \frac{3}{4}$

Need $\lambda \ge 0$ which is ok when $\frac{a-3}{4a-3} \ge 0 \qquad a-3 \ge 0 \qquad a \ge 3$

$\Rightarrow$ Optimal when $a \ge 3$

Problem 3-03

a) max $x + xy$ s.t. $y + x^2 e^y \leq 1$

$\mathcal{L} = x + xy - \lambda(y + x^2 e^y - 1)$

(1) $\frac{\partial \mathcal{L}}{\partial x} = 1 + y - 2\lambda x e^y = 0$

(2) $\frac{\partial \mathcal{L}}{\partial y} = x - \lambda(1 + x^2 e^y) = 0$

(3) $\lambda \geq 0$ ($\lambda = 0$ if $y + x^2 e^y < 1$)

(4) $y + x^2 e^y \leq 1$

b) • $(0,-1)$:

(4) gives $-1 < 1$ ok! (4) is satisfied, ($\Rightarrow \lambda = 0$)

(1) gives $0 = 0$ ok!

(2) gives $\lambda = 0$ ok! (1)-(4) all satisfied $\lambda = 0$

• $(1,0)$:

(4) gives $1 \leq 1$ ok!

(1) gives $1 = 2\lambda$ $\boxed{\lambda = \frac{1}{2}}$ ok!

(2) gives $\lambda(1+1) = 1$ $\Rightarrow \lambda = \frac{1}{2}$ ok!

(1) -(4) all satisfied $\lambda = \frac{1}{2}$

$\boxed{\text{Problem 3-05}}$

$$\max\ x^2 y e^{-x-y} \quad \text{s.t.} \begin{cases} x \geq 1 \\ y \geq 1 \\ x+y \geq 4 \end{cases}$$

a) $\mathcal{L} = x^2 y e^{-x-y} - \lambda_1(1-x) - \lambda_2(1-y) - \lambda_3(4-x-y)$

(1) $\dfrac{\partial \mathcal{L}}{\partial x} = 2xye^{-x-y} - x^2 y e^{-x-y} + \lambda_1 + \lambda_3 = 0$

(2) $\dfrac{\partial \mathcal{L}}{\partial y} = x^2 e^{-x-y} - x^2 y e^{-x-y} + \lambda_2 + \lambda_3 = 0$

(3) $\lambda_1 \geq 0 \quad (\lambda_1 = 0 \text{ if } x > 1)$

(4) $\lambda_2 \geq 0 \quad (\lambda_2 = 0 \text{ if } y > 1)$

(5) $\lambda_3 \geq 0 \quad (\lambda_3 = 0 \text{ if } x+y > 4)$

(6) $x \geq 1$

(7) $y \geq 1$

(8) $x+y \geq 4$

b) ~~binding~~ ~~binding~~

zero binding

~~$\Rightarrow x=1\ y=1$~~ • Non binding $\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

$\Rightarrow$ (2) gives $\underline{y=1}$ ~~gives~~ ~~...~~ Impossible

↳ Not ok since (7) not binding

1 binding

• (6) binding, (7), (8) nonbinding $\Rightarrow \underline{\lambda_2 = \lambda_3 = 0}$, $\underline{x=1}$

(2) gives $\underline{y=1}$ (1) gives $\lambda_1 = -e^{-2} < 0$ $\boxed{\text{Impossible}}$

• (7) only binding $\Rightarrow \underline{\lambda_1 = \lambda_3 = 0}$, $\underline{y=1}$

(1) gives $\underline{x=2}$ (2) gives $\underline{\lambda_2 = 0}$

That is $\boxed{(x,y) = (2,1)}$ $\boxed{\text{Impossible}}$ (Since $x+y \geq 4$).

• only (8) binding $\Rightarrow$ $\lambda_2 = \lambda_1 = 0$ $\quad x+y=4$

(1) gives $\lambda_3 = xye^{-4}[x-2]$

(2) gives $x^2 e^{-4}(1-y) = (2-x)xye^{-4} \Rightarrow x(1-y) = (2-x)y$

$\Rightarrow x - xy = 2y - xy \quad x = 2y \Rightarrow 3y = 4 \quad y = \frac{4}{3}$

$\Rightarrow x = 4 - \frac{4}{3} = \frac{8}{3} \quad x = \frac{8}{3} \Rightarrow \lambda_3 = \frac{32}{9}e^{-4}\left(\frac{2}{3}\right) > 0$ ok.

so $\boxed{(x,y) = \left(\frac{8}{3}, \frac{4}{3}\right) \text{ is candidate}}$

2 binding

• $x=1 \quad y=1 \quad x+y>4$ $\boxed{\text{impossible}}$

• $x=1 \quad y>1 \quad x+y=4 \quad \lambda_2 = 0 \Rightarrow y=3$

(2) gives $\lambda_3 = 2e^{-4} > 0$ $\quad$ (1) gives $\lambda_1 = -2e^{-4} - 3e^{-4} < 0$ $\boxed{\text{impossible}}$

• $x>1 \quad y=1 \quad x+y=4 \quad \lambda_1 = 0 \Rightarrow x=3$

(1) gives $\lambda_3 = 9e^{-4} - 6e^{-4} = 3e^{-4} > 0$

(2) gives $\lambda_2 = -3e^{-4}$ $\boxed{\text{impossible}}$

3 binding $\quad x=1 \quad y=1 \quad x+y=4$ $\boxed{\text{impossible}}$

~~c) Yes, the only candidate was $(x,y) = (\frac{8}{3}, \frac{4}{3})$~~

c) Yes. Note that as either x or $y \to \infty$, $x^2ye^{-x-y} \to 0$ as e^{-x-y} dominates (exponential). Also, $x^2ye^{-x-y} > 0$ for any specific allowed values of x and y. Thus, there must be some points where the function reaches a max value, and since we only have one candidate, it must be it.

$\boxed{\text{Problem 3-12}}$

a) $\max\ -(x+\frac{1}{2})^2-\frac{1}{2}y^2 \quad \text{s.t.} \begin{cases} y \geq e^{-x} \\ y \leq \frac{2}{3} \end{cases}$

$$\mathcal{L}(x,y) = -(x+\tfrac{1}{2})^2 - \tfrac{1}{2}y^2 - \lambda(e^{-x}-y) - \mu(y-\tfrac{2}{3})$$

$$\boxed{\begin{array}{ll}
(1) & \frac{\partial \mathcal{L}}{\partial x} = -2(x+\frac{1}{2}) + \lambda e^{-x} = 0 \\
(2) & \frac{\partial \mathcal{L}}{\partial y} = -y + \lambda - \mu = 0 \\
(3) & \lambda \geq 0 \quad (\lambda = 0 \text{ if } y > e^{-x}) \\
(4) & \mu \geq 0 \quad (\mu = 0 \text{ if } y < \frac{2}{3}) \\
(5) & y \geq e^{-x} \\
(6) & y \leq \frac{2}{3}
\end{array}}$$

• Guess $y = e^{-x}$ and $\underline{y=\frac{2}{3}}$ $\Rightarrow e^{-x} = \frac{2}{3}$ $\quad -x = \ln\frac{2}{3}$ $\quad \underline{x = \ln\frac{3}{2}}$

(1) gives $\lambda = 2(\ln\frac{3}{2}+\frac{1}{2})\cdot\frac{3}{2} > 0$

(2) gives $\mu = \lambda - y = 3(\ln\frac{3}{2}+\frac{1}{2}) - \frac{2}{3} > 0$ ok! $\Rightarrow \boxed{(x,y) = (\ln\frac{3}{2}, \frac{2}{3}) \text{ is candidate}}$

• Guess $\underline{y = e^{-x}}$ $\quad y < \frac{2}{3}$ $\Rightarrow \underline{\mu = 0}$

(2) gives $\underline{\lambda = e^{-x}}$ (1) gives $-2(x+\frac{1}{2}) + e^{-2x} = 0$ $\quad e^{-2x} = 2x+1$ $\Rightarrow \underline{x=0}$

$\Rightarrow \underline{y=1}$ $\boxed{\text{Impossible}}$ since $y < \frac{2}{3}$

• Guess $y > e^{-x}$, $\underline{y = \frac{2}{3}}$ $\Rightarrow \underline{\lambda = 0}$ $\quad$ (2) gives $\mu = -\frac{2}{3} < 0$ $\boxed{\text{Impossible}}$

• Guess $y > e^{-x}$ $\quad y < \frac{2}{3}$ $\Rightarrow \underline{\lambda = \mu = 0}$ $\quad \Rightarrow$ (2) gives $\underline{y = 0}$

$\Rightarrow y = 0 > e^{-x}$ Impossible since $e^{-x} > 0$ for all x. $\boxed{\text{Impossible}}$

Since only one candidate, the solution is $\underline{(x,y) = (\ln\frac{3}{2}, \frac{2}{3})}$

b) $y \geq e^{-x} \Rightarrow \underline{y > 0}$ and $y \leq \frac{2}{3} \Rightarrow \underline{x > 0}$ (or else $e^{-x} > 1 > \frac{2}{3}$).

Want x close to $-\frac{1}{2}$ and y close to 0, but cost of missing target of y half that of x
$\Rightarrow$ Most important to get x as small as possible, which occur when $y = e^{-x}$, and y large as possible, i.e. $y = \frac{2}{3}$

Problem 3-14

NB!

a) $\min \, (x-2)^2+(y-2)^2$ s.t. $\begin{cases} x+y\le 2 \\ x^2-4x+y\le -2\end{cases}$ equivalent to $\max \, -(x-2)^2-(y-2)^2$ given same constraints

$$\mathcal{L} = -(x-2)^2 - (y-2)^2 - \lambda(x+y-2) - \mu(x^2-4x+y+2)$$

$$\boxed{\begin{array}{ll}
(1) & \frac{\partial \mathcal{L}}{\partial x} = -2(x-2)-\lambda-2\mu x+4\mu = 0 \\
(2) & \frac{\partial \mathcal{L}}{\partial y} = -2(y-2)-\lambda-\mu = 0 \\
(3) & \lambda \ge 0 \quad (\lambda > 0 \text{ if } x+y<2) \\
(4) & \mu \ge 0 \quad (\mu > 0 \text{ if } x^2-4x+y<-2) \\
(5) & x+y \le 2 \\
(6) & x^2-4x+y \le -2
\end{array}}$$

- Non binding $\Rightarrow \lambda = \mu = 0$

$\Rightarrow$ (1) gives $x=2$ (2) gives $y=2$ contradicts (5) $\boxed{\text{Impossible}}$

- 1 binding

- $x+y=2$ $\quad x^2-4x+y<-2 \Rightarrow \mu=0$

(1) gives $\lambda = -2(x-2)$ (2) gives $2(y-2) = 2(x-2) \Rightarrow x=y$

$\Rightarrow \boxed{x=y=1}$ But this gives (6) binding, a contradiction $\boxed{\text{Impossible}}$

- $x+y<2$ $\quad x^2-4x+y=-2$ $\quad \lambda=0$

(2) gives $\mu=-2(y-2)$ (1) gives $-2(x-2)+4x(y-2)-8(y-2)=0$

$\Rightarrow (x-2)+(y-2)(4-2x)=0$ $\quad (x-2)=2(y-2)(x-2)$

- If $x\ne 2$ then $2(y-2)=1 \Rightarrow y=\frac{1}{2}+2$ $\quad y=\frac{5}{2}$ $\Rightarrow x^2-4x=-\frac{9}{2}$

$x^2-4x+\frac{9}{2}=0$ No solution

- If $x=2$ then $0=0$ ok. $\Rightarrow 4-8+y=-2$ $\quad y=2$ Contradicts $x+y<2$ $\boxed{\text{Impossible}}$

• All binding $\Rightarrow$ $x+y=2$ $\quad x^2-4x+y=-2$

$\Rightarrow \underline{y=2-x} \quad \Rightarrow x^2-4x+2-x=-2$

$\Rightarrow x^2-5x+4=0 \quad \Rightarrow \underline{x=4}$ or $\underline{x=1}$

$x=4 \Rightarrow y=-2 \qquad x=1 \Rightarrow y=1$

Two candidates $(x_1,y_1)=(4,-2)$ and $(x_2,y_2)=(1,1)$.

~~• (4,-2) gives in (2) λ = -8-μ ≥ 0 if μ ≥ 0 impossible~~

~~• (1,1) gives in (2) λ = -2+μ < 0 if μ ≥ 0 impossible~~

• $(4,-2)$ gives in (2) : $\underline{\lambda = 8-\mu}$

(1) gives $-4-8+\mu-8\mu+4\mu=0 \quad 3\mu=-12<0$ $\boxed{\text{Impossible}}$

• $(1,1)$ gives in (2) $\underline{\lambda = 2-\mu}$

(1) gives $2+\mu-2-2\mu+4\mu=0 \Rightarrow 3\mu=0 \quad \underline{\mu=0} \Rightarrow \underline{\lambda=2>0}$

ok! $\Rightarrow$ $\boxed{(x,y)=(1,1) \text{ is candidate}}$

Since the only candidate, $\underline{(x,y)=(1,1) \text{ is optimal solution}}$

b) It's about finding the minimal distance from the point $(x,y)=(2,2)$ given the constraints. If you draw the two constraints ($y \le 2-x$ and $y \le -x^2+4x-2$), then you will see that $(1,1)$ is the point among all points satisfying the constraints that is closest to the point $(2,2)$.