Problem 3-02

$$\max \{x^2+y^2\} \text{ s.t. } 5x^2+6xy+5y^2 \le 1$$

a) $\mathcal{L}(x,y) = x^2+y^2-\lambda(5x^2+6xy+5y^2-1)$

$$\boxed{\begin{array}{l} (1)\ \frac{\partial \mathcal{L}}{\partial x} = 2x-10\lambda x-6\lambda y = 0 \\ (2)\ \frac{\partial \mathcal{L}}{\partial y} = 2y-10\lambda y-6\lambda x = 0 \\ (3)\ \lambda \ge 0 \quad (\lambda = 0 \text{ if } 5x^2+6xy+5y^2<1) \\ (4)\ 5x^2+6xy+5y^2 \le 1 \end{array}}$$

- if (4) is binding, then (1) gives $2\lambda(5x+3y)=2x$

  If $5x+3y \neq 0$, then $\boxed{\lambda = \frac{x}{5x+3y}}$

  If $5x+3y=0$, then $x=0$, which means $y^2=\frac{1}{5}$, but this is not possible since $5x+3y=0$. Thus we can assume $5x+3y\neq 0$.

  (2) then gives (dividing by 2) $y-(5y+3x)\frac{x}{5x+3y}=0$

  $\Rightarrow \quad (5x+3y)y = (3x+5y)x \Rightarrow \boxed{y^2=x^2} \Rightarrow \boxed{y=\pm x}$

  Then (4) gives two possibilities:

  $5x^2+6x^2+5x^2 = 16x^2 = 1$ or $5x^2-6x^2+5x^2 = 4x^2 = 1$

  $\Rightarrow \quad x=\pm\frac{1}{4}$ with $y=x$ or $x=\pm\frac{1}{2}$ with $y=-x$.

  ~~Since λ≥0 must hold,~~ All combinations satisfy $\lambda \ge 0$.

  Thus, we have four candidates:

$$\boxed{\begin{array}{l} (x,y)=(\frac{1}{4},\frac{1}{4}), \\ (x,y)=(-\frac{1}{4},-\frac{1}{4}) \\ (x,y)=(\frac{1}{2},-\frac{1}{2}) \\ (x,y)=(-\frac{1}{2},\frac{1}{2}) \end{array}}$$

- if (4) is non-binding, then $\boxed{\lambda = 0}$.

Then (1) and (2) gives $\boxed{x=0}$ and $\boxed{y=0}$.

This does not contradict (4).

$\boxed{\text{Thus, } (x,y) = (0,0) \text{ is a candidate}}$

The best candidates are $(x,y) = (\frac{1}{2}, -\frac{1}{2})$ and $(x,y) = (-\frac{1}{2}, \frac{1}{2})$, since this gives highest value x^2+y^2.

Both gives $\boxed{\lambda = \frac{1}{2}}$.

Then, $\mathcal{L}''_{xx} = 2 - 10\lambda = 2-5 = -3 < 0$

$\mathcal{L}''_{yy} = 2 - 10\lambda = -3 < 0$

$\mathcal{L}''_{xy} = -6\lambda = -3 \Rightarrow \mathcal{L}''_{xx}\mathcal{L}''_{yy} - (\mathcal{L}''_{xy})^2 = (-3)(-3) - (-3)^2 = 0$

Thus, $\mathcal{L}$ is concave, and we therefor conclude that <u>both $(x,y) = (\frac{1}{2}, -\frac{1}{2})$ and $(x,y) = (-\frac{1}{2}, \frac{1}{2})$ solve the problem.</u>

b) We want x^2+y^2 as great as possible, thus we want to be as far away from the origin as possible.
$5x^2+6xy+5y^2 \leq 1$ makes the relevant ellipse that restrict our domain

c) $\Delta V \approx \lambda \cdot 0{,}1 = \frac{1}{2} \cdot 0{,}1 = \underline{0{,}05}$

Answers to problems about concave and quasiconcave programming.

- One-line proof that the admissible set for a concave programming problem is convex:

  By definition, the constraint functions $g(x) \le c$ is convex, and therefore quasiconvex. By definition of quasiconvexity, the admissible set is therefore convex.
  (intersection of convex sets are convex, so it is true also for many g_j).

- f quasiconcave, g_j quasiconvex.

  Again the admissible set is convex, since each g_j is quasiconvex and therefore makes convex sets, and the intersection of all these will again be convex.