$\boxed{6-10\ a}$

$$\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - \rho V(x) = w - x \qquad (*)$$

$V(x) = x^a \qquad V'(x) = a x^{a-1} \qquad V''(x) = a(a-1) x^{a-2}$

Incerted in homogeneous version of $(*)$ gives

$$\frac{1}{2}\sigma^2 a(a-1) x^a + \mu a x^a - \rho x^a = 0$$

$\Rightarrow$ a must solve $\boxed{\frac{1}{2}\sigma^2 a(a-1) + \mu a - \rho = 0}$

Solution is $\quad a_{1,2} = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\underbrace{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}_{>0 \text{ since } \rho > 0}} \qquad (1)$

Thus there exists a such that x^a satisfies the problem (homogeneous version).

And since we have two such a's, x^{a_1} and x^{a_2} will be linear independent, and we then know linear combination of these will make the general solution,

Thus, the general solution is $\underline{\underline{V(x) = A x^{a_1} + B x^{a_2}}}$,
where $a_{1,2}$ is as in (1).

Problem 6-12

$$\dot{p}(t) = \beta \int_{-\infty}^{t} [a+c-(b+d)p(\tau)] e^{-\gamma(t-\tau)} d\tau$$

a) $\ddot{p} = \beta\left[a+c-(b+d)p(t) - \gamma\int_{-\infty}^{t}(a+c-(b+d)p(\tau))e^{-\gamma(t-\tau)}d\tau\right]$

$$= \beta(a+c) - \beta(b+d)p - \gamma\dot{p}$$

$$\Rightarrow \underline{\ddot{p} + \gamma\dot{p} + \beta(b+d)p = \beta(a+c)}$$

b) $\dot{p} = 0$ and $\ddot{p} = 0 \Rightarrow \underline{p^* = \frac{a+c}{b+d}}$

General solution: $\frac{1}{4}\gamma^2 - \beta(b+d) \begin{cases} >0 & \text{case I} \\ =0 & \text{case II} \\ <0 & \text{case III} \end{cases}$

Case I: $p = Ae^{r_1 t} + Be^{r_2 t}$ where $r_{1,2} = -\frac{1}{2}\gamma \pm \sqrt{\frac{1}{4}\gamma^2 - \beta(b+d)}$

Case II: $p = (A+Bt)e^{-\frac{1}{2}\gamma t}$

Case III: $p = e^{-\frac{1}{2}\gamma t}(A\cos \theta t + B\sin \theta t)$ where $\theta = \sqrt{\beta(b+d) - \frac{1}{4}\gamma^2}$

To these general solution is $p + v^*$, when $v^* = p^* \Rightarrow p + \frac{a+c}{b+d}$

c) Since $\gamma > 0$, $-\frac{1}{2}\gamma < 0$, and also note that both $r_1 < 0$ and $r_2 < 0$

Thus, case I $\lim_{t\to\infty}(p + p^*) = p^*$ since $\lim_{t\to\infty} p = 0$

case II $\lim_{t\to\infty} (A+Bt)e^{-\frac{1}{2}\gamma t} + p^* = p^*$ (even though $Bt \to \infty$, the exponent will dominate!)

case III $\lim_{t\to\infty}(p + p^*) = p^*$ ~~will fluctuate in the interval~~ ...

$\Rightarrow$ Stable in all cases

In the case $\frac{1}{4}\gamma^2 - \beta(b+d) < 0$ we have the fluctuating case due to sin, cos functions.