ECON 4140 - Seminar 8

$\boxed{\text{Problem 7-01}}$

$\dot{x} = x + y + t \quad (1)$

$\dot{y} = -x + 2y \quad (2)$

$\cancel{\ddot{x} = \dot{x} + \dot{y} + 1 = \dot{x} - x + 2y + 1}$

(2) gives $x = 2y - \dot{y} \Rightarrow \dot{x} = 2\dot{y} - \ddot{y}$

Thus, (1) becomes $2\dot{y} - \ddot{y} = 2y - \dot{y} + y + t$ i.e. $\underline{\ddot{y} - 3\dot{y} + 3y = -t}$ (*)

$$\frac{1}{4}(-3)^2 - 3 = \frac{9}{4} - 3 = \frac{9-12}{4} < 0$$

$\Rightarrow x = e^{\frac{3}{2}t}(A\cos \beta t + B \sin \beta t)$ where $\beta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$\underline{x = e^{\frac{3}{2}t}\left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right)}$

$u^* = At + B \quad \dot{u}^* = A \quad \ddot{u}^* = 0$

$\Rightarrow$ inserted in (*) gives $-3A + 3At + 3B = -t \Rightarrow 3A = -1 \quad \boxed{A = -\frac{1}{3}}$

$-3A + 3B = 0 \qquad 3B = -1 \quad \boxed{B = -\frac{1}{3}} \Rightarrow \underline{u^* = -\frac{1}{3}t - \frac{1}{3}}$

$\Rightarrow$ General solution of x: $\underline{\underline{x = e^{\frac{3}{2}t}\left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right) - \frac{1}{3}t - \frac{1}{3}}}$

$\Rightarrow \dot{x} = \frac{3}{2}e^{\frac{3}{2}t}\left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right) + e^{\frac{3}{2}t}\left(-\frac{\sqrt{3}}{2}A\sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}B\cos\left(\frac{\sqrt{3}}{2}t\right)\right) - \frac{1}{3}$

$\Rightarrow y = \dot{x} - x - t = \underline{\underline{\frac{1}{2}e^{\frac{3}{2}t}\left[A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right] + e^{\frac{3}{2}t}\left[\frac{\sqrt{3}}{2}B\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}A\sin\left(\frac{\sqrt{3}}{2}t\right)\right] - \frac{2}{3}t}}$

Problem 7-04 a & b

$\dot{x} = y - x^2 - xy = f(x,y)$
$\dot{y} = x - y^2 - xy = g(x,y)$

a) $\dot{x} = 0 \Rightarrow y - x^2 - xy = 0 \Rightarrow xy = y - x^2$
$\dot{y} = 0 \Rightarrow x - y^2 - xy = 0$

$x - y^2 - y + x^2 = 0 \qquad x + x^2 = y + y^2 \Rightarrow \underline{x = y}$

$\Rightarrow x - 2x^2 = 0 \qquad x^2 - \frac{1}{2}x = 0 \qquad x(x - \frac{1}{2}) = 0 \qquad x = 0$ or $x = \frac{1}{2}$

$\Rightarrow \boxed{(x,y) = (0,0) \text{ or } (x,y) = (\tfrac{1}{2}, \tfrac{1}{2})}$

~~0 = y - x^2 - xy partial derivative w.r.t. x gives 0 = -2x - y~~

$f_1' = -2x - y \qquad f_2' = 1 - x$

$g_1' = 1 - y \qquad g_2' = -2y - x$

$A(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow tr(A(0,0)) = 0, \ |A(0,0)| = -1 < 0$

Since the determinant is negative, $\underline{(0,0) \text{ is saddle point}}$

$A(\frac{1}{2},\frac{1}{2}) = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \Rightarrow tr(A(\frac{1}{2},\frac{1}{2})) = -3 < 0 \qquad |A(\frac{1}{2},\frac{1}{2})| = \frac{9}{4} - \frac{1}{4} = 2 > 0$

Since determinant positive and trace negative, $\underline{(\frac{1}{2},\frac{1}{2}) \text{ is asymptotically stable}}$

b) $z = x + y \qquad \dot{z} = \dot{x} + \dot{y} = y - x^2 - xy + x - y^2 - xy = x + y - (x^2 + 2xy + y^2) = z - z^2$

$\Rightarrow \underline{\dot{z} = z - z^2}$ ~~Bernoulli form~~ i.e. $\underline{\dot{z} - z = -z^2}$ Bernoulli form

$\Rightarrow w = z^{-1} \qquad \Phi - \dot{w} - w = -1 \qquad \dot{w} + w = 1 \qquad \frac{d}{dt}(e^t w) = e^t \qquad e^t w = e^t + C$

~~$w = ...$~~ $w = 1 + Ce^{-t} \Rightarrow z = \frac{1}{1 + Ce^{-t}} \qquad \underline{z = \frac{e^t}{e^t + C}}$

$\boxed{\text{Exam 2008, Problem 1}}$

a) $C_k = \begin{pmatrix} k & 2 & k \\ 2 & 3 & 0 \\ k & 0 & k \end{pmatrix}$ $\quad D_k = \begin{pmatrix} k & 2 & k & k \\ 2 & 3 & 0 & k+1 \\ k & 0 & k & ke^k \end{pmatrix}$

$|C_k| = k\begin{vmatrix} 3 & 0 \\ 0 & k \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ k & k \end{vmatrix} + k\begin{vmatrix} 2 & 3 \\ k & 0 \end{vmatrix} = 3k^2 - 4k - 3k^2 = -4k$

$\boxed{k \neq 0 \Rightarrow \text{Rank}(C_k) = 3}$ $\Rightarrow \boxed{\text{Rank}(D_k) = 3 \text{ for } k \neq 0}$

$\boxed{k = 0 \Rightarrow \text{Rank}(C_0) = 2}$

$k = 0 \Rightarrow D_0 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\Rightarrow \boxed{\text{Rank}(D_0) = 2}$

Has solution for all $k \neq 0$

b) $A = C_0 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

i) $\begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \begin{pmatrix} 2y \\ 2x+3y \\ 0 \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$ $\quad \Rightarrow z = 0 \quad y = 2x$

$2x + 6x = 8x$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

ii) $\begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \lambda_2\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\lambda_2 \\ -\lambda_2 \\ 0 \end{pmatrix}$ $\quad \Rightarrow \lambda_2 = -1$

iii) ~~trace A~~ $\text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 3$

$4 - 1 + \lambda_3 = 3 \quad \lambda_3 = 0$

d) i) $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y \\ 2x+3y \\ 0 \end{pmatrix}$

$\dot{z} = 0 \Rightarrow z = C$ $\quad \dot{x} = 2y \quad y = \frac{1}{2}\dot{x} \quad \dot{y} = \frac{1}{2}\ddot{x}$

$\dot{y} = 2x + 3y \Rightarrow \frac{1}{2}\ddot{x} = 2x + \frac{3}{2}\dot{x} \Rightarrow \ddot{x} - 3\dot{x} - 4x = 0$

$\frac{1}{4}9 + 4 > 0 \Rightarrow x = Ae^{4t} + Be^{-t}$ $\quad y = 2Ae^{4t} - \frac{1}{2}Be^{-t}$

$\frac{9+16}{4} = \frac{25}{4}$

~~ii) $x(0) = A + B = 0 \Rightarrow A = -B$~~
~~$y(0) = 2A - \frac{1}{2}B = -2B - \frac{1}{2}B = 0$ ... $-4B - B = 0$ ... $B = 0 \Rightarrow$~~

ii) We need $\lim_{t\to\infty} x = 0$ and $\lim_{t\to\infty} y = 0$

Only possible when $\boxed{A = 0}$

Thus $x^* = Be^{-t}$ $\quad y^* = -\frac{1}{2}Be^{-t}$ $\Rightarrow$ $\underline{\frac{x^*}{y^*} = -2}$ $\quad$ (Here, $B \neq 0$).

$\boxed{\text{Induction exercise}}$

a) $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

- Base case $0 + 1 = 1$ $\quad \frac{1(1+1)}{2} = \frac{2}{2} = 1$ ok for base case

- Assume true for n, is then true for $n+1$?

$$0 + 1 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Yes, is true for $n+1$ as well $\Rightarrow$ <u>Hypothesis true</u>

b) • $\frac{9-1}{8} = \frac{8}{8} = 1$ $\Rightarrow$ Possible for base case.

- Assume true for n, is true for $n+1$? This means $\frac{9^n - 1}{8} = k$ ← integer

~~$\frac{9^{n+1}}{8} = \frac{9 \cdot 9^n}{8} - \frac{9^n}{8} \cdots \frac{1}{8}$~~ or $9^n - 1 = 8k$ or $\underline{9^n = 8k + 1}$

Then, $\frac{9^{n+1} - 1}{8} = \frac{9 \cdot 9^n - 1}{8} = \frac{9 \cdot (8k+1) - 1}{8} = \frac{9 \cdot 8k + 9 - 1}{8}$

$= 9k + \frac{9}{8} - \frac{1}{8} = 9k + \frac{8}{8} = 9k + 1$ which is integer since k integer.

$\Rightarrow$ <u>Hypothesis true</u>

$\boxed{7-02}$

$$\dot{x} = ax + 2y + \alpha \qquad (1)$$
$$\dot{y} = 2x + ay + \beta \qquad (2)$$

a) (1) gives $\underline{y = \frac{1}{2}\dot{x} - \frac{a}{2}x - \frac{\alpha}{2}}$ $\Rightarrow$ $\underline{\dot{y} = \frac{1}{2}\ddot{x} - \frac{a}{2}\dot{x}}$

incert in (2): $\frac{1}{2}\ddot{x} - \frac{a}{2}\dot{x} = 2x + \frac{a}{2}\dot{x} - \frac{a^2}{2}x - \frac{\alpha a}{2} + \beta$

$$\Rightarrow \boxed{\ddot{x} - 2a\dot{x} + (a^2-4)x = 2\beta - \alpha a}$$

"Characteristic coefficient": $\frac{1}{4}(-2a)^2 - (a^2-4) = 4 > 0$

~~$\Rightarrow x(t) = Ae^{2at} + Be$~~

$$\Rightarrow x^*(t) = Ae^{(a+2)t} + Be^{(a-2)t}$$

When $a \neq \pm 2$, then $\underline{u^*(t) = \frac{2\beta - \alpha a}{a^2-4}}$

When $a = 2$, $\underline{u^*(t) = \lambda t + \mu}$ $\Rightarrow u^{*\prime}(t) = \lambda$, $u^{*\prime\prime}(t) = 0$

Thus, $u^*(t)$ must satisfy ~~$2a\lambda + (a^2-4)(\lambda t + \mu)$~~

$$-4\lambda = 2\beta - \alpha a \Rightarrow \lambda = \frac{2a - 2\beta}{4} = \underline{\frac{1}{2}(\alpha - \beta)}$$

When $a = -2$, $u^*(t) = \bar{\lambda} t + \bar{\mu}$ $\Rightarrow u^{*\prime}(t) = \bar{\lambda}$, $u^{*\prime\prime}(t) = 0$

$$\Rightarrow 4\bar{\lambda} = 2(\beta + \alpha) \Rightarrow \underline{\bar{\lambda} = \frac{1}{2}(\alpha + \beta)}$$

To summarize, for x we have the following

$$\boxed{\begin{aligned}
&\text{When } a \neq \pm 2, \quad x(t) = Ae^{(a+2)t} + Be^{(a-2)t} + \frac{2\beta - \gamma a}{a^2-4} \\
&\text{When } a = 2, \quad x(t) = Ae^{4t} + \tfrac{1}{2}(\gamma - \beta)t + \mu \\
&\text{When } a = -2, \quad x(t) = Be^{-4t} + \tfrac{1}{2}(\gamma + \beta)t + \bar{\mu}
\end{aligned}} \qquad (*)$$

We know $y(t) = \frac{1}{2}\dot{x} - \frac{a}{2}x - \frac{\gamma}{2}$,

and
$$\dot{x} = \begin{cases} (a+2)Ae^{(a+2)t} + (a-2)Be^{(a-2)t} & \text{when } a \neq \pm 2 \\ 4Ae^{4t} + \frac{1}{2}(\gamma - \beta) & \text{when } a = 2 \\ -4Be^{-4t} + \frac{1}{2}(\gamma + \beta) & \text{when } a = -2 \end{cases}$$

Thus,
$$y(t) = \begin{cases} Ae^{(a+2)t} \; Be^{(a-2)t} + \dfrac{\frac{\gamma a^2}{2} - \beta a}{a^2-4} - \dfrac{\frac{\gamma}{2}(a^2-4)}{(a^2-4)} & \text{when } a \neq \pm 2 \\ (2 - \frac{a}{2})Ae^{4t} - \frac{a}{4}(\gamma - \beta)t - \frac{a}{2}\mu + \frac{1}{4}(\gamma - \beta) - \frac{\gamma}{2} & \text{when } a = 2 \\ -(2 + \frac{a}{2})Be^{-4t} - \frac{a}{4}(\gamma + \beta)t - \frac{a}{2}\bar{\mu} + \frac{1}{4}(\gamma + \beta) - \frac{\gamma}{2} & \text{when } a = -2 \end{cases}$$

Thus
$$\boxed{y(t) = \begin{cases} Ae^{(a+2)t} - Be^{(a-2)t} + \dfrac{2\gamma - \beta a}{(a^2-4)} & \text{when } a \neq \pm 2 \\ Ae^{4t} - \frac{1}{2}(\gamma - \beta)t - \mu - \frac{1}{4}(\alpha + \beta) & \text{when } a = 2 \\ -Be^{-4t} + \frac{1}{2}(\gamma + \beta)t + \bar{\mu} + \frac{1}{4}(\beta - \gamma) & \text{when } a = -2 \end{cases}} \qquad (**)$$

<u>$(*)$ and $(**)$ give the solution of the problem</u>

b) $\dot{x} = 0$ and $\dot{y} = 0$

$$\Rightarrow \begin{array}{l} ax + 2y + \alpha = 0 \\ 2x + ay + \beta = 0 \end{array} \Rightarrow \overbrace{\begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}$$

Since $a \neq \pm 2$, $\begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$ is invertible $\left(\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} = a^2 - 4 \neq 0 \right)$.

$\Rightarrow$ Solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}^{-1} \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}$.

$$\begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}^{-1} = \frac{1}{a^2 - 4} \begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4 - a^2} \begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{4 - a^2} \begin{pmatrix} a\alpha - 2\beta \\ a\beta - 2\alpha \end{pmatrix}$$

i.e. $x = \dfrac{a\alpha - 2\beta}{4 - a^2}$ and $y = \dfrac{a\beta - 2\alpha}{4 - a^2}$ in equilibrium.

Locally asymptotic stable iff globally a.s.

$\Leftrightarrow$ $tr(A) < 0$ and $|A| > 0$.

$tr\, A = 2a$ and $|A| = a^2 - 4$

Thus, the system is locally asymptotically stable if and only if $a < -2$

when $|A| < 0$, we have a saddle point.

Thus, we have a saddle point when $a \in (-2, 2)$