Seminar 1 (Matte 3)

Oppgave 1

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

$$vAv^T = (\overbrace{a\ b\ c}^{\neq 0}) \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (4a+b \quad a+2c \quad 2b+4c) \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= (4a+b)a + (a+2c)b + (2b+4c)c = 4a^2 + 2ab + 4bc + 4c^2$$

kan anta både positive og negative verdier

(Feks $a = c = 1$, $b = 1$ gir positiv
$a = c = 1$, $b = -10$ gir negativ)

Er ingen av delene

Oppgave 2

a) $f(\bar{x}) = Ax_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ $\quad a_i > 0$

Hessian $H = \begin{pmatrix} f''_{11} & f''_{12} & \cdots & f''_{1n} \\ f''_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ f''_{n1} & \cdots & \cdots & f''_{nn} \end{pmatrix}$

$f'_i = a_i A x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_n^{a_n}$ $\qquad f''_{ii} = a_i(a_i - 1) A x_1^{a_1} \cdots x_i^{a_i - 2} \cdots x_n^{a_n} = \underline{\dfrac{a_i(a_i-1) f(x)}{x_i^2}}$

$= \underline{\dfrac{a_i f(x)}{x_i}}$ $\qquad f''_{ij} = a_i a_j A x_1^{a_1} \cdots x_i^{a_i-1} \cdots x_j^{a_j-1} \cdots x_n^{a_n} = \underline{\dfrac{a_i a_j f(x)}{x_i x_j}}$

$\Rightarrow$ ~~H~~ H matrisen over der $f_{ii} = \dfrac{a_i(a_i-1)f(x)}{x_i^2}$ og $f_{ij} = \dfrac{a_i a_j f(x)}{x_i x_j}$

b) $n = 3$, $a_1 + a_2 + a_3 < 1$

principal subdeterminant

f konkav $\Leftrightarrow (-1)^r \Delta_r(x) \geq 0 \quad \forall x \in \mathbb{R}^3$ og $\forall \Delta_r$, $r = 1, 2, 3$

$r = 1$; $-1 \cdot |f_{ii}| = -\dfrac{a_i(a_i-1)f(x)}{x_i^2} > 0$

$r = 2$: $(-1)^2 \begin{vmatrix} f_{ii} & f_{ij} \\ f_{ji} & f_{jj} \end{vmatrix} = f_{ii} f_{jj} - f_{ij}^2 = \dfrac{a_i a_j (a_i - 1)(a_j - 1)[f(x)]^2}{x_i^2 x_j^2} - \dfrac{a_i^2 a_j^2 [f(x)]^2}{x_i^2 x_j^2}$

$= \dfrac{[f(x)]^2}{x_i^2 x_j^2} a_i a_j \left((a_i - 1)(a_j - 1) - a_i a_j\right) = \dfrac{[f(x)]^2}{x_i^2 x_j^2} a_i a_j (1 - a_i - a_j) > 0$

$$r=3: \quad (-1)^3 \begin{vmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{vmatrix} = -\left(\underbrace{f''_{11} \begin{vmatrix} f''_{22} & f''_{23} \\ f''_{32} & f''_{33} \end{vmatrix}}_{+} \overset{+\text{(from previous)}}{} - f''_{12} \begin{vmatrix} f''_{21} & f''_{23} \\ f''_{31} & f''_{33} \end{vmatrix} + f''_{13} \begin{vmatrix} f''_{21} & f''_{22} \\ f''_{31} & f''_{32} \end{vmatrix} \right)$$

$$= -f''_{11}\Delta_2 + (f''_{12})^2 f''_{33} - f''_{12} f''_{13} f''_{23} - f''_{12} f''_{13} f''_{23} + f''_{22}(f''_{13})^2$$

$$= [f(x)]^3 \left(-\frac{a_1(a_1-1)}{x_1^2} \frac{a_2 a_3(1-a_2-a_3)}{x_2^2 x_3^2} + \frac{a_1^2 a_2^2}{x_1^2 x_2^2} \frac{a_3(a_3-1)}{x_3^2} - 2\frac{a_1^2 a_2^2 a_3^2}{x_1^2 x_2^2 x_3^2} + \frac{a_2(a_2-1)}{x_2^2} \frac{a_1^2 a_3^2}{x_1^2 x_3^2} \right)$$

$$= \frac{[f(x)]^3}{x_1^2 x_2^2 x_3^2} a_1 a_2 a_3 \left[\underbrace{-(a_1-1)(1-a_2-a_3) + a_1 a_2(a_3-1) - 2a_1 a_2 a_3 + (a_2-1) a_1 a_3} \right]$$

$$= -a_1(1-a_2-a_3) + 1 - a_2 - a_3 + a_1 \cancel{a_2 a_3} - a_1 a_2 - \cancel{2a_1 a_2 a_3} + \cancel{a_1 a_2 a_3} - a_1 a_3$$

$$= -a_1 + \cancel{a_1 a_2} + \cancel{a_1 a_3} + 1 - a_2 - a_3 - \cancel{a_1 a_2} - \cancel{a_1 a_3} = 1 - a_1 - a_2 - a_3 \;\underline{> 0}$$

$(-1)^r \Delta_r(x) > 0 \quad \forall x \in \mathbb{R}^3_+,\ \forall \Delta_r \Rightarrow$ strictly Concave

Without differentiating:

~~Concave if $f(\lambda\bar{x} + (1-\lambda)\bar{y}) \geq \lambda f(\bar{x}) + (1-\lambda) f(\bar{y}) \quad \forall \bar{x}, \bar{y} \in \mathbb{R}^3_+,\ \forall \lambda \in (0,1)$~~

~~$A(\lambda x_1 + (1-\lambda)y_1)^{a_1}(\lambda x_2 + (1-\lambda)y_2)^{a_2}(\lambda x_3 + (1-\lambda)y_3)^{a_3} \geq \lambda A x_1^{a_1} x_2^{a_2} x_3^{a_3} + (1-\lambda) A y_1^{a_1} y_2^{a_2} y_3^{a_3}$~~

~~$(\lambda x + (1-\lambda)y)^a \geq \lambda x^a + (1-\lambda)y^a$~~

$z = A x_1^{a_1} x_2^{a_2} x_3^{a_3} \qquad \ln z = \ln A + a_1 \ln x_1 + a_2 \ln x_2 + a_3 \ln x_3$ sum av konkave funksjoner $\Rightarrow$ konkav

$z = e^{\ln A + a_1 \ln x_1 + a_2 \ln x_2 + a_3 \ln x_3}$

Problem 2-01

a) streng Konkavitet følger av forrige oppgave for (i) $\Rightarrow$ kvasikonkav

(ii) $f''_{11} < 0$ $f''_{22} > 0$ Hverken konveks eller konkav, men kvasikonkav pga

$\ln A + a_1 \ln x_1 + a_2 \ln x_2$ kvasikonkav (pga konkav) $\rightarrow e^z$ voksende

b) $f'_1 = -2x_1 - x_2 + x_1$ $f'_2 = -x_1 - 4x_2$ $f'_3 = x_1 - 10x_3$

$$H = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -4 & 0 \\ 1 & 0 & -10 \end{pmatrix}$$

$f''_{ii} < 0$ $\Rightarrow -f''_{11} > 0$

$\Delta_2 > 0$

$$\Delta_3 = -\begin{vmatrix} -4 & 0 \\ 0 & -10 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 1 & -10 \end{vmatrix} = -40 + 10 = -30$$

$\Rightarrow -1\Delta_3 > 0$ $\Rightarrow$ strictly Concave $\Rightarrow$ Quasiconcave

c) Concave, since $x+y-4$ concave (and convex) $\Rightarrow \ln(x+y-4)$ concave since $\ln(\cdot)$ concave and growing

$\Rightarrow \sqrt{\ln(\cdot)}$ concave since $\sqrt{\cdot}$ concave and growing

Concave $\Rightarrow$ Quasiconcave   Defined for $x+y>4$

d) $H'_1 = 6x_1 - 2x_2 - 4x_3$ $H'_2 = -2x_1 + 2x_2$ $H'_3 = -4x_1 + 4x_3$

$$\mathcal{H} = \begin{pmatrix} 6 & -2 & -4 \\ -2 & 2 & 0 \\ -4 & 6 & 4 \end{pmatrix}$$

$\Delta_1 = 6, 2, 4$ $\Delta_1 > 0$

$$\Delta_2 = \begin{vmatrix} 6 & -2 \\ -2 & 2 \end{vmatrix} = 12 - 4 = 8 \quad \begin{vmatrix} 2 & 0 \\ 6 & 4 \end{vmatrix} = 8 \quad \Delta_2 > 0$$

$$\Delta_3 = 6\begin{vmatrix} 2 & 0 \\ 6 & 4 \end{vmatrix} + 2\begin{vmatrix} -2 & 0 \\ -4 & 4 \end{vmatrix} - 4\begin{vmatrix} -2 & 2 \\ -4 & 6 \end{vmatrix} = 48 - 16 + 16 = 48 \quad \Delta_3 > 0$$

strictly Convex $\Rightarrow$ Quasiconvex   Since all leading principal minors are positive, the function is positive definite

~~Minimum H. F.o.c.~~ ~~$H'_2 = 0 \Rightarrow x_1 = x_2$~~ ~~$H'_3 = 0 \Rightarrow x_1 = x_3$~~

H'_1

$$(a\ b\ c)\begin{pmatrix} 6 & -2 & -4 \\ -2 & 2 & 0 \\ -4 & 6 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (6a - 2b - 4c \quad -2a + 2b + 6c \quad -4a + 4c)\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= 6a^2 - 2ab - 4ac - 2ab + 2b^2 + 6bc - 4ac + 4c^2 = 6a^2 + 2b^2 + 4c^2 - 4ab - 8ac + 6bc$$

Problem 2-02

$f(x,y) = e^{ax+by^2}$

e^z growing convex $\Rightarrow$ $f(x,y)$ convex if $ax+by^2$ convex,
which it is for all $b \geq 0$ (a doesn't matter)

$f_x' = a e^{ax+by^2}$ $f_y' = 2by e^{ax+by^2}$

$$H = \begin{pmatrix} a^2 e^{ax+by^2} & 2aby e^{ax+by^2} \\ 2aby e^{ax+by^2} & 2b e^{ax+by^2} + 4b^2y^2 e^{ax+by^2} \end{pmatrix} = \begin{pmatrix} a^2 f & 2abyf \\ 2abyf & 2bf + 4b^2y^2 f \end{pmatrix}$$

$a^2 f \geq 0$ $2bf(1+2by^2) \geq 0$ if $b \geq 0$ ($b<0$ not possible $\forall y$)

not possible with $-2bf(1+2by^2) \geq 0$ $\forall y$, for then $b \leq 0$, but the paranthesis expression would take both positive and negative values depending on y.

$\Rightarrow$ Only convex, $\forall b \geq 0$ $b<0$ ~~neither~~ neither convex or concave

Problem 2-03

$f(x,y) = 2x - y - e^x - e^{x+2y}$ is concave

because $2x-y$ is both convex and concave, $-e^x$ is concave and decreasing, while x and $x+2y$ convex (and concave)
$\Rightarrow$ $-e^x$ and $-e^{x+2y}$ concave, sum of concave functions is concave,

(in theory should check that also is not ~~convex~~ convex, but dont need to as can quickly see that any double derivative is non-zero $\Rightarrow$ can't be convex and concave simultaniously).

$\boxed{\text{Problem 2-06}}$

$f(x,y) = \frac{1}{12}(x-y)^4 - (x-y)^2 - (x+y)^2 \qquad x \in (-1,1) \quad y \in (-1,1)$

a) $f'_x = \frac{1}{3}(x-y)^3 - 2(x-y) - 2(x+y) \qquad f'_y = -\frac{1}{3}(x-y)^3 + (x-y) - (x+y)$

$$H = \begin{pmatrix} (x-y)^2 - 4 & -(x-y)^2 \\ -(x-y)^2 & (x-y)^2 - 2 \end{pmatrix}$$

~~$(x-y)^2 - 2 \geq 0 \quad (x-y)^2 \geq 2$~~

convex: $(x-y)^2 - 4 \geq 0 \qquad (x-y)^2 \geq 4 \qquad x-y \geq 2$ and $x-y \leq -2$ impossible since $x \in (-1,1)$ and $y \in (-1,1)$ (not including ± 1).

• concave if $-(x-y)^2 + 4 \geq 0 \qquad 4 \geq (x-y)^2$ OK.

and if $-(x-y)^2 + 2 \geq 0 \qquad 2 \geq (x-y)^2$ <u>not ok in general</u>,

b) but ok if $x - y \leq \sqrt{2}$ and $x-y \geq -\sqrt{2}$

$|H| = ((x-y)^2 - 4)((x-y)^2 - 2) - (x-y)^4 = (x-y)^4 - 6(x-y)^2 + 8 - (x-y)^4 = 8 - 6(x-y)^2 \geq 0$

$\Rightarrow \quad 4 \geq 3(x-y)^2 \qquad (x-y)^2 \leq \frac{4}{3} \qquad x-y \leq \frac{2}{\sqrt{3}}$ and $x-y \geq -\frac{2}{\sqrt{3}}$

$< \sqrt{2}$

$\Rightarrow$ <u>Concave over $-\frac{2}{\sqrt{3}} \leq x-y \leq \frac{2}{\sqrt{3}}$</u>

$\boxed{\text{Problem 2-07}}$

$f(x,y) = (\ln x)^a (\ln y)^b \qquad x>1,\ y>1,\ a>0,\ b>0,\ a+b<1$

$f'_x = a(\ln x)^{a-1}\frac{1}{x}(\ln y)^b \qquad f'_y = b(\ln x)^a(\ln y)^{b-1}\frac{1}{y}$

$$H = \begin{pmatrix} a(a-1)(\ln x)^{a-2}\frac{1}{x}(\ln y)^b - a(\ln x)^{a-1}\frac{1}{x^2}(\ln y)^b & \frac{ab(\ln x)^{a-1}(\ln y)^{b-1}}{xy} \\ \frac{ab(\ln x)^{a-1}(\ln y)^{b-1}}{xy} & \frac{b(b-1)(\ln x)^a(\ln y)^{b-2}}{y} - \frac{b(\ln x)^a(\ln y)^{b-1}}{y^2} \end{pmatrix}$$

$-D_1 = \frac{-a(a-1)(\ln x)^{a-2}(\ln y)^b}{x} + \frac{a(\ln x)^{a-1}(\ln y)^b}{x^2} > 0$

$\frac{(\ln x)^{a-1}(\ln y)^b}{x^2} > \frac{(a-1)(\ln x)^{a-2}(\ln y)^b}{x} \qquad \frac{1}{x} > (a-1)(\ln x)^{-1}$

$\frac{\ln x}{x} > a-1$ ok siden $a-1<0$ og $\frac{\ln x}{x}>0$

$D_2 = \frac{a(\ln x)^{a-1}(\ln y)^b}{x}\left[\frac{(a-1)}{\ln x} - \frac{1}{x}\right] \cdot \frac{b(\ln y)^{b-1}(\ln x)^a}{y}\left[\frac{(b-1)}{\ln y} - \frac{1}{y}\right] - \frac{a^2b^2(\ln x)^{2a-2}(\ln y)^{2b-2}}{x^2y^2} > 0$

$\left[\frac{a-1}{\ln x} - \frac{1}{x}\right]\left[\frac{(b-1)}{\ln y} - \frac{1}{y}\right] > \frac{ab(\ln x)^{-1}(\ln y)^{-1}}{xy} = \frac{ab}{xy\ln x\ln y}$

$\left(\frac{(a-1)x-\ln x}{x\ln x}\right)\left(\frac{(b-1)y-\ln y}{y\ln y}\right) > \frac{ab}{xy\ln x\ln y} \qquad [(a-1)x-\ln x][(b-1)y-\ln y] > ab$

$(ax-x-\ln x)(by-y-\ln y) > ab \qquad abxy - axy - ax\ln y - bxy + xy + x\ln y - b\ln x\, y + y\ln x + \ln x\ln y > ab$

$abxy + xy + x\ln y + y\ln x + \ln x\ln y - axy - bxy - ax\ln y - by\ln x - ab > 0$

$= \underset{>0}{ab(xy-1)} + \underset{>0}{xy(1-a-b)} + \underset{>0}{x\ln y(1-a)} + \underset{>0}{y\ln x(1-b)} + \underset{>0}{\ln x\ln y} > 0$ OK!

$\Rightarrow$ Strictly concave