Seminar 11 - ECON 4140

Problem 9-02

$$\max_{u \in \mathbb{R}} \int_0^T -(x-u+2)^2 e^{-rt}\,dt \quad , \quad \dot{x} = u(t) - \delta x(t) \quad , \quad x(0) = x_0 \quad , \quad x(T) = x_T$$

$T, r, \delta, x_0, x_T > 0.$

a) $H(t,x,u,p) = -(x-u+2)^2 e^{-rt} + p(u - \delta x)$

- u maximizes $H(t,x^*,u,p)$
- $\dot{p} = -H^*_x$
- $p(T)$ free

b) ~~$H'_x = e^{-rt} \delta p \quad H''_{xx} = 0 \quad H'_u$~~

$H'_x = -2(x-u+2)e^{-rt} - p\delta \qquad H''_{xx} = -2e^{-rt} < 0$

$H'_u = 2(x-u+2)e^{-rt} + p \qquad H''_{uu} = -2e^{-rt} < 0$

$H''_{xu} = 0 \quad \Rightarrow H''_{xx}H''_{uu} - (H''_{xu})^2 = 4e^{-2rt} > 0 \quad \Rightarrow$ Concave in (x,u)

c) $H(t,x,u,p) = -(x-u+2)^2 e^{-0,1t} + p(u - 0,5x)$

- u: u maximizes $H(t,x^*,u,p)$ #1
- p: $\dot{p} = 2(x-u+2)e^{-0,1t} + 0,5p$ #2; $p(T)$ free
- x: $\dot{x} = u - 0,5x$ #3; $x(0) = 0$ #4; $x(10) = 8$ #5

H concave in u $\Rightarrow$ (#1) can use first order condition to find u:

$2(x-u+2)e^{-0,1t} + p = 0$

$\Rightarrow u = x + 2 + \frac{1}{2}e^{0,1t}p$

(#2) $\Rightarrow \dot{p} = 2\left(-\frac{1}{2}e^{0,1t}p\right)e^{-0,1t} + 0,5p = -0,5p$

$\dot{p} + 0,5p = 0 \qquad \frac{d}{dt}(pe^{0,5t}) = 0$

$p = Ce^{-0,5t}$

(#3): $\dot{x} = x + 2 + \frac{1}{2}e^{0,1t} C e^{-0,5t} - 0,5x = 0,5x + 2 + \frac{1}{2} C e^{-0,4t}$

$\dot{x} - 0,5x = 2 + \frac{1}{2} C e^{-0,4t}$ $\quad \frac{d}{dt}\left(x e^{-0,5t}\right) = 2e^{-0,5t} + \frac{1}{2} C e^{-0,9t}$

$x e^{-0,5t} = -4e^{-0,5t} - \frac{5}{9} C e^{-0,9t} + D$

$x(t) = -4 - \frac{5}{9} C e^{-0,4t} + D e^{0,5t}$

(#4) $x(0) = -4 - \frac{5}{9} C + D = 0$ $\quad D = 4 + \frac{5}{9} C$

(#5) $x(10) = -4 - \frac{5}{9} C e^{-4} + (4 + \frac{5}{9} C) e^{5} = 8$

$\frac{5}{9} C [e^{5} - e^{-4}] = 12 - 4e^{5}$ $\quad \frac{5}{9} C = \frac{12 - 4e^{5}}{e^{5} - e^{-4}}$ $\quad D = 4 + \frac{12 - 4e^{5}}{e^{5} - e^{-4}} = \frac{12 - 4e^{-4}}{e^{5} - e^{-4}}$

$$x^*(t) = -4 + \frac{4e^{5} - 12}{e^{5} - e^{-4}} e^{-0,4t} + \frac{12 - 4e^{-4}}{e^{5} - e^{-4}} e^{0,5t}$$

$$p(t) = \frac{9}{5} \frac{12 - 4e^{5}}{e^{5} - e^{-4}} e^{-0,5t}$$ $$u(t) = x(t) + 2 + \frac{1}{2} e^{0,1t} p(t)$$

Problem 9-03

a) $\max_{u \in \mathbb{R}} \int_0^2 (x(t) - u(t)^2)\,dt$, $\dot{x}(t) = x(t) + u(t)$, $x(0) = 0$, $x(2)$ free

$H(t,x,u,p) = x - u^2 + p(x+u)$ $\quad H'_x = 1 + p$

$$\begin{array}{lll} u & u \in \mathbb{R} \text{ maximizes } H(t, x^*, u, p) & (1) \\ p & \dot{p} = -H_x & (2) \\ & p(2) = 0 & (3) \\ x & \dot{x} = x + u & (4) \\ & x(0) = 0 & (5) \end{array}$$

H concave in u ⇒ can use F.O.C.

(1) $\frac{dH}{du} = -2u + p = 0 \Rightarrow \underline{u = \frac{1}{2}p}$

(2) $\dot{p} = -1 - p \Rightarrow \dot{p} + p = -1 \quad \frac{d}{dt}(pe^t) = -e^t \quad pe^t = -e^t + C$

$\underline{p(t) = Ce^{-t} - 1}$

(3): $p(2) = Ce^{-2} - 1 = 0 \quad \underline{C = e^2} \Rightarrow \boxed{p(t) = e^{2-t} - 1}$

$\Rightarrow \boxed{u(t) = \frac{1}{2}(e^{2-t} - 1)}$

(4): $\dot{x} = x + \frac{1}{2}(e^{2-t} - 1) \quad \dot{x} - x = \frac{1}{2}(e^{2-t} - 1) \quad \frac{d}{dt}(xe^{-t}) = \frac{1}{2}(e^{2-2t} - e^{-t})$

$xe^{-t} = -\frac{1}{4}e^{2-2t} + \frac{1}{2}e^{-t} + D \quad \underline{x(t) = -\frac{1}{4}e^{2-t} + De^t + \frac{1}{2}}$

(5): $x(0) = -\frac{1}{4}e^2 + D + \frac{1}{2} = 0 \quad \underline{D = \frac{1}{4}e^2 - \frac{1}{2}}$

$\Rightarrow \boxed{x^*(t) = -\frac{1}{4}e^{2-t} + \left(\frac{1}{4}e^2 - \frac{1}{2}\right)e^t + \frac{1}{2}}$

$x - u^2$ concave in (x,u) ⇒ know candidate is optimal solution.

b) Here, (1) might change.

From a), $u(t) = \frac{1}{2}(e^{2-t} - 1)$

we need to concider if this might not be in $[0,1]$ for $t \in [0,2]$

Checks $u(0) = \frac{1}{2}(e^{2} - 1) \approx 3{,}1945 > 1$ problem!

Ask: at what time t^* does $u(t)$ start to be in $[0,1]$?

$u(t^*) = \frac{1}{2}(e^{2-t^*} - 1) = 1$ $\quad e^{2-t^*} = 3$ $\quad 2 - t^* = \ln 3$ $\quad \underline{t^* = 2 - \ln 3} \approx 0{,}9$

i.e. for $t \in [2 - \ln 3, 2]$ F.O.C. will give $u(t) \in [0,1]$ i.e. $u(t) = \frac{1}{2}(e^{2-t} - 1)$.

For $t \in [0, 2 - \ln 3)$ we want $u(t)$ as close to this as possible, i.e.
choose corner solution $u(t) = 1$.

Thus, $u(t) = \begin{cases} \frac{1}{2}(e^{2-t} - 1) & t \in [2 - \ln 3, 2] \\ 1 & t \in [0, 2 - \ln 3) \end{cases}$

In the case $u = 1$ we get of (4): $\dot{x} = x + 1 \Rightarrow \frac{d}{dt}(x e^{-t}) = e^{-t}$

$\Rightarrow x e^{-t} = -e^{-t} + C$ $\quad x = Ce^{t} - 1$ $\quad x(0) = C - 1 = 0 \Rightarrow \underline{C = 1}$

$\Rightarrow \underline{x(t) = e^{t} - 1}$

~~Thus, $x(t) = \begin{cases} \text{as in part a) for } t \in [2 - \ln 3, 2] \\ e^{t} - 1 \text{ for } t \in [0, 2 - \ln 3) \end{cases}$~~

After time $t = 2 - \ln 3$, $x(t) = -\frac{1}{4}e^{2-t} + De^{t} + \frac{1}{2}$, but constant D is determined now by the continuity of $x(t)$. This means the two x's derived must be equal at time $2 - \ln 3$. Thus: $e^{2-\ln 3} - 1 = \underbrace{-\frac{1}{4}e^{\ln 3}}_{= -\frac{3}{4}} + De^{2-\ln 3} + \frac{1}{2}$

$e^{2-\ln 3} - \frac{3}{4} = De^{2-\ln 3}$

$D = 1 - \frac{3}{4}e^{\ln 3 - 2} = \underline{1 - \frac{9}{4}e^{-2}}$

$$\Rightarrow \boxed{x(t) = \begin{cases} e^{t} - 1 & \text{for } t \in [0, 2 - \ln 3 \\ -\frac{1}{4}e^{2-t} + (1 - \frac{9}{4}e^{-2})e^{t} + \frac{1}{2} & \text{for } t \in (2 - \ln 3, 2] \end{cases}}$$

Problem 9-05

$$\max \int_0^T (ax^2 + 2bx\dot{x} + c\dot{x}^2 + dt^2\dot{x}) e^{-rt}\, dt \quad , x(0) = x_0 \, , x(T) = x_T$$

a) $f(x,y) = ax^2 + 2bxy + cy^2 + dt^2 y$

$\frac{\partial f}{\partial x} = 2ax + 2by \quad \frac{\partial f}{\partial y} = 2bx + 2cy + dt^2$

$\frac{\partial^2 f}{\partial x^2} = 2a \quad \frac{\partial^2 f}{\partial x \partial y} = 2b \quad \frac{\partial^2 f}{\partial y^2} = 2c$

Need $a \le 0$, $b \le 0$, $4ac - 4b^2 \ge 0$ $\quad ac \ge b^2$

Thus concave if $\boxed{a \le 0, c \le 0, \; ac \ge b^2}$ $\boxed{r \text{ doesn't matter}}$

b) $\frac{\partial F}{\partial x} = (2ax + 2b\dot{x}) e^{-rt} \quad \frac{\partial F}{\partial \dot{x}} = (2bx + 2c\dot{x} + dt^2) e^{-rt}$

$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = (2b\dot{x} + 2c\ddot{x} + 2dt) e^{-rt} - r(2bx + 2c\dot{x} + dt^2) e^{-rt}$

Thus, $\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2e^{-rt}\left[ax + b\dot{x} + r(bx + c\dot{x} + \frac{1}{2}dt^2) - b\dot{x} - c\ddot{x} - dt\right] = 0$

$$\boxed{\ddot{x} - r\dot{x} - \frac{(a+rb)}{c}x = \frac{1}{2}\frac{r}{c}dt^2 - \frac{d}{c}t}$$ Euler equation

c) $a = -9$, $b = 1$, $c = -1$, $d = 3$, $x_0 = 0$, $x_T = 0$, $T = 1$, $r = 0$ $\Rightarrow$ Problem concave

~~Euler gives~~ Euler eq. gives $\boxed{\ddot{x} - 9x = 3t}$

$\Rightarrow x(t) = Ae^{r_1 t} + Be^{r_2 t} + u^*$ where $r_{1,2} = \pm 3$

$u^* = Ct + D \quad \Rightarrow \dot{u}^* = C \quad \ddot{u}^* = 0$

$\Rightarrow -9(Ct + D) = 3t \quad \Rightarrow \underline{C = -\frac{1}{3}} \quad \underline{D = 0} \quad x(t) = Ae^{3t} + Be^{-3t} - \frac{1}{3}t$

$x(0) = A + B = 0 \quad \underline{B = -A} \quad x(1) = A\left[e^{3} - e^{-3}\right] - \frac{1}{3} = 0 \quad A = \frac{1}{3(e^{3} - e^{-3})}$

$$x(t) = \frac{e^{3t}}{3(e^3 - e^{-3})} - \frac{e^{-3t}}{3(e^3-e^{-3})} - \frac{1}{3}t$$

d) $\max_{u \in \mathbb{R}} \int_0^1 (-9x^2 + 2xu - u^2 + 3t^2 u)\,dt$, $\dot{x} = u$, $x(0) = 0$, (i) $x(1)$ free, (ii) $x(1) \geq 2$

$$H(t,x,u,p) = -9x^2 + 2xu - u^2 + 3t^2 u + pu$$

(1) u maximizes $H(t,x^*,u,p)$

(2) $\dot{p} = -H_x = 18x - 2u$

(3) $p(1) \begin{cases} = 0 & \text{(i)} \\ \geq 0 \ (p(1) = 0 \text{ if } x(1) > 2) & \text{(ii)} \end{cases}$

(4) $\dot{x} = u$

(5) $x(0) = 0$

(6) $x(1)$ free (i)
$x(1) \geq 2$ (ii)

OBS! This part is not necessary, one knows that the Euler equation must still hold in the transformed problem. I show this formally here just for convenience. You might skip right to (*)

(1) gives F.o.c. $2x - 2u + 3t^2 + p = 0$ $\quad u = x + \frac{3}{2}t^2 + \frac{1}{2}p$

$H'_x = -18x + 2u$

(2) gives $\dot{p} = 18x - 2u = 18x - 2x - 3t^2 - p$ $\quad \dot{p} + p = 16x - 3t^2$

By (4) we thus have
$$\begin{cases} \dot{x} = x + \frac{3}{2}t^2 + \frac{1}{2}p \\ \dot{p} = -p - 3t^2 + 16x \end{cases}$$
system of differential equations

We get $p = 2\dot{x} - 2x - 3t^2$ $\quad \dot{p} = 2\ddot{x} - 2\dot{x} - 6t$

Incert in below equation: $2\ddot{x} - 2\dot{x} - 6t = -2\dot{x} + 2x + 3t^2 - 3t^2 + 16x$

$\boxed{\ddot{x} - 9x = 3t}$ The Euler equation! (*)

$\Rightarrow x(t) = Ae^{3t} + Be^{-3t} - \frac{1}{3}t$ $\quad x(0) = A + B = 0$ $\quad B = -A$

$x(t) = A(e^{3t} - e^{-3t}) - \frac{1}{3}t$

$$p(t) = -\frac{\partial F}{\partial \dot{x}} = -\frac{\partial F}{\partial u} = -2x + 2\dot{x} - 3t^2$$

$$p = -2x + 2\left(A(3e^{3t}+3e^{-3t}) - \frac{1}{3}\right) - 3t^2 = \underline{-2x + 6A(e^{3t}+e^{-3t}) - \frac{2}{3} - 3t^2}$$

(i) $x(1)$ free $\Rightarrow$ $p(1) = 0$

$$p(1) = -2x(1) + 6A(e^{3}+e^{-3}) - \frac{2}{3} - 3$$

$$= -2A(e^3 - e^{-3}) + \frac{2}{3} + 6A(e^3+e^{-3}) - \frac{2}{3} - 3 = 0$$

$$A[4e^3 + 8e^{-3}] = 3 \qquad \underline{A = +\frac{3}{4(e^3+2e^{-3})}} \qquad \underline{B = \frac{-3}{4(e^3+2e^{-3})}}$$

$$\Rightarrow \underline{x(t) = +\frac{3}{4(e^3+2e^{-3})}(e^{3t}-e^{-3t}) - \frac{1}{3}t} \qquad (\mathrm{I})$$

(ii) $x(1) \geq 2$

- case 1 $x(1) = 2$ $\Rightarrow$ $x(1) = A(e^3 - e^{-3}) - \frac{1}{3} = 2$ $\quad A = \frac{7}{3(e^3-e^{-3})}$

$$\Rightarrow \underline{x(t) = \frac{7}{3(e^3-e^{-3})}(e^{3t}-e^{-3t}) - \frac{1}{3}t} \qquad (\mathrm{II})$$

- case 2 $x(1) > 2$ $\Rightarrow$ $p(1) = 0$ $\Rightarrow$ gets (I) $\Rightarrow$ $x(1) < 1$, a contradiction

  Thus one chooses such that $x(1) = 2$ and arrives at (II)

To summarize:

With (i) one gets (I)
With (ii) one gets (II)

Then $u(t)$ follows by $u = \dot{x}$

Problem 9-07

$$\max_{u \in [0,1]} \int_0^T (1-u)x^2 \, dt \quad \dot{x} = ux \;, \; x(0)=1 \quad x(T) \text{ free}$$

a) $H(t,x,u,p) = (1-u)x^2 + pux \qquad H'_x = 2(1-u)x + pu$

#1 u^* maximizes $(1-u)x^2 + pux$

#2 $\begin{cases} \dot{p} = -2(1-u)x - pu \\ p(T) = 0 \end{cases}$

#3 $\begin{cases} \dot{x} = ux \\ x(0) = 1 \end{cases}$

b) $\dot{x} = ux$ since $u \in [0,1]$ and $x(0)=1$ we get that $\dot{x} \geq 0$

$\Rightarrow x(t) \geq 1$

$\dot{p} = \underbrace{-2\overbrace{(1-u)}^{\geq 0} \overset{\geq 1}{x}}_{\leq 0} - p\overset{\geq 0}{u}$ $\quad \dot{p}(T) = -2(1-u(T))x(T) \leq 0$. Thus $p(t) \geq 0$ must be true close to time T but then we see that $\dot{p} \leq 0$ is true here as well, and will thus always need to be true $\Rightarrow p(t) \geq 0$.

~~$\Rightarrow p(t)$ must be ≥ 0 close to T $\Rightarrow \dot{p} = -2(1-u)x - pu \geq 0$ close to T~~

Assume $p(t) = 0$ for any $t < T$. Then $\dot{p} = -2(1-u)\overset{\geq 1}{x}$ must equal zero, or we will forever get negative p. That means $u=1$, but that contradicts #1.

Thus $p(t) > 0$ for all $t < T$. Thus, since $x \geq 1$ ~~we see that~~ and $u \in [0,1]$ we see that $\dot{p} < 0$ must be

c) We know $p(t) \geq 0$ for all times, and is strictly decreasing.

- If $p(0) \leq 1$ $p(t) \leq 1$ for all times $\Rightarrow u = 0$ always since $x(t) \geq 1$.

Then we get $\dot{p} = -2x$ and $\dot{x} = 0$ $\Rightarrow \dot{p} = -2$ $p(t) = -2t + C$

$p(T) = 0 \Rightarrow -2T + C = 0$ $C = 2T$ $p(t) = 2(T-t)$

$\Rightarrow p(0) = 2T > 2 \cdot \frac{1}{2} = 1$ A contradiction!

- Thus $p(0) > 1$ must hold. Then we know we start by $u = 1$ because of #1. Since $p(T) = 0$ and $x(t) \geq 1$ there is a time t^* when $p(t^*) = x(t^*)$. After that time $p(t) < x(t)$ since $\dot{p} < 0$, $\dot{x} \geq 0$
$\Rightarrow$ change strategy to $u = 0$ (because of #1). Must find this t^*,

$$u(t) = \begin{cases} 1 & t \in [0, t^*] \\ 0 & t \in (t^*, T] \end{cases} \Rightarrow \dot{x} = \begin{cases} x & t \in [0, t^*] \\ 0 & t \in (t^*, T] \end{cases}$$

$$\Rightarrow x(t) = \begin{cases} e^t & t \in [0, t^*] \\ D & t \in (t^*, T] \end{cases} \quad x(0) = 1 \Rightarrow x(t) = e^t$$

Must be equal at time t^* (since $x(t)$ continuous)

$$\Rightarrow x(t^*) = e^{t^*} = D$$

$$\dot{p} = \begin{cases} -p & t \in [0, t^*] \Rightarrow p(t) = Ee^{-t} \\ -2D & t \in (t^*, T] \Rightarrow p(t) = -2Dt + F = -2e^{t^*}t + F \end{cases}$$

$p(T) = -2e^{t^*}T + F = 0$ $F = 2e^{t^*}T$ $\Rightarrow$ $p(t) = 2e^{t^*}(T-t)$ for $t \in (t^*, T]$

$p(t^*) = Ee^{-t^*} = 2e^{t^*}(T-t^*)$ $\Rightarrow$ $E = 2e^{2t^*}(T-t^*)$ $\Rightarrow$ $p(t) = 2e^{2t^*}(T-t^*)e^{-t}$ for $t \in [0, t^*]$

$x(t^*) = p(t^*)$ by definition of t^* $\Rightarrow 2e^{t^*}(T-t^*) = e^{t^*}$ $\Rightarrow T - t^* = \frac{1}{2}$ $t^* = T - \frac{1}{2}$

Thus, (x, u) and p is as above with $t^* = T - \frac{1}{2}$.

Problem 9-10

$$\max_{u\in[0,\infty)} \int_0^1 -(x-u-a)^2\,dt \qquad \dot{x}=u-x \qquad x(0)=1 \qquad x(1) \text{ free}$$

a) $H(t,x,u,p) = -(x-u-a)^2 + p(u-x)$ $\qquad H'_x = -2(x-u-a)-p$

(1) { u^* maximizes $H(t,x,u,p)$

(2) { $\dot{p} = 2(x-u-a)+p$

(3) { $p(1)=0$

(4) { $\dot{x} = u-x$

(5) { $x(0)=1$

(1) gives F.o.c. $2(x-u-a)+p=0$ $\qquad u = x-a+\frac{1}{2}p$

(2) $\dot{p} = 2(x-x+a-\frac{1}{2}p-a)+p = 0 \Rightarrow p(t)=C$

(3) $p(1)=0 \Rightarrow \boxed{p(t)=0} \Rightarrow$ ~~...~~ $u=x-a$

(4) $\dot{x} = x-a-x = -a \Rightarrow x(t) = -at+D$

(5) $x(0) = D = 1 \Rightarrow \boxed{x(t)=1-at}$

$\Rightarrow \boxed{u(t) = 1-a(t+1)}$

$a=4 \Rightarrow u(t) = 1-4t-4 = -4t-3 < 0 \Rightarrow$ Choose corner solution $u=0$

(Remember $u\in[0,\infty)$!).

H is concave for all a because $-z^2$ is concave and decreasing and $x-u-a$ is convex ~~and $p(u-x)$ is concave~~ $\Rightarrow -(x-u-a)^2$ concave by result of concavity; $p(u-x)$ linear and thus concave $\Rightarrow H$ concave. Mangasarian ok!

b) $a = \frac{1}{3} \Rightarrow u(t) = 1 - \frac{1}{3}t - \frac{1}{3} = \frac{2}{3} - \frac{1}{3}t$

$u(t) \geq 0$ when $\frac{2}{3} \geq \frac{1}{3}t$ i.e. when $2 \geq t$ Always holds since $t \in [0,1]$

~~$\Rightarrow u(t) = 1 - \frac{1}{3}(t+1)$~~ $\Rightarrow$ $u(t) = \frac{2}{3} - \frac{1}{3}t$

Problem 9-17

a) $\max_{u \in \mathbb{R}} \int_0^1 x + u \, dt \quad \dot{x} = 1 - \frac{1}{2}u^2 \quad x(0) = 0 \quad x(1) \geq 0$

$H(t, x, u, p) = x + u + p(1 - \frac{1}{2}u^2) \qquad H'_x = 1$

(1) u^* maximizes $x + u + p(1 - \frac{1}{2}u^2)$

(2) $\dot{p} = -1$

(3) $p(1) \geq 0$ ($p(1) = 0$ if $x(1) > 0$)

(4) $\dot{x} = 1 - \frac{1}{2}u^2$

(5) $x(0) = 0$

(6) $x(1) \geq 0$

(2) gives $p = -t + C$

(1) gives F.O.C. $1 - pu = 0 \Rightarrow u = \frac{1}{p}$ $\quad u = \frac{1}{C - t}$

(4) gives $\dot{x} = 1 - \frac{1}{2(C-t)^2}$ $\quad x(t) = t - \frac{1}{2(C-t)} + D$

(5) gives $x(0) = -\frac{1}{2C} + D = 0$ $\quad D = \frac{1}{2C}$ $\Rightarrow$ $x(t) = t - \frac{1}{2(C-t)} + \frac{1}{2C}$

- Assume $x(1) = 0 \Rightarrow x(1) = 1 - \frac{1}{2(C-1)} + \frac{1}{2C} = 0 \Rightarrow 2C(C-1) - 2C + 2(C-1) = 0$

$4C^2 - 4C - 2 = 0 \quad C^2 - C - \frac{1}{2} = 0 \quad C = \frac{1 \pm \sqrt{1+2}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

Since $p(1) \geq 0$ must have $C = \frac{1}{2} + \frac{\sqrt{3}}{2}$

thus $x(t) = t - \frac{1}{1+\sqrt{3}-2t} + \frac{1}{1+\sqrt{3}}$, $u(t) = \frac{1}{\frac{1}{2}+\frac{\sqrt{3}}{2}-t}$

$p(t) = \frac{1}{2} + \frac{\sqrt{3}}{2} - t$

- Assume $x(1) > 0 \Rightarrow p(1) = 0 \Rightarrow C - 1 = 0 \Rightarrow C = 1$

$\Rightarrow p(t) = 1 - t$

$\Rightarrow u(t) = \frac{1}{1-t}$ $\quad x(t) = t - \frac{1}{2(1-t)} + \frac{1}{2}$

$\Rightarrow \lim_{t \to 1} x(t) = -\infty$ impossible!

$\Rightarrow$ The case with $x(1) = 0$ occurs

$\Rightarrow$ $x(t) = t - \frac{1}{1+\sqrt{3}-2t} + \frac{1}{1+\sqrt{3}}$ $\quad u(t) = \frac{1}{\frac{1}{2}+\frac{\sqrt{3}}{2}-t}$ $\quad p(t) = \frac{1}{2} + \frac{\sqrt{3}}{2} - t$

b) Here, $U(0) = \frac{2}{1+\sqrt{3}} < 1$ $\quad U(1) = \frac{2}{\sqrt{3}-1} > 1$

$\Rightarrow$ There is a t^* when $U(t^*) = 1$:

$\frac{1}{2} + \frac{\sqrt{3}}{2} - t^* = 1$ $\quad t^* = \frac{\sqrt{3}-1}{2}$ $\quad$ For $t \in [t^*, 1]$ $U(t) = \frac{1}{\frac{1}{2}+\frac{\sqrt{3}}{2}-t}$

For $t \in [0, t^*]$ choose corner solution $u = 1$

$\Rightarrow \dot{x} = \frac{1}{2}$ $\quad x = \frac{1}{2}t + C$ $\quad x(0) = 0 \Rightarrow x(t) = \frac{1}{2}t$

$p = -t + D$ $\quad p(t^*) = D - \frac{\sqrt{3}-1}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2} - \left(\frac{\sqrt{3}-1}{2}\right)$

$\Rightarrow D = \frac{1}{2} + \frac{\sqrt{3}}{2} \Rightarrow p(t) = \frac{1}{2} + \frac{\sqrt{3}}{2} - t$ for all $t \in [0, t^*]$

x

b) Here, $u(0) = \frac{2}{2+\sqrt{3}} < 1$, $u(1) = \frac{2}{\sqrt{3}-1} > 1$

$\Rightarrow$ There is a t^* when $u(t^*) = 1$.

- In $t \in [0, t^*]$ will choose corner solution $u(t) = 1$

Thus, for $t \in [0, t^*]$ $u(t) = 1$, $\dot{x} = \frac{1}{2}$ $\Rightarrow$ $x(t) = \frac{1}{2}t + E$ $x(0) = 0 \Rightarrow E = 0$

$\Rightarrow$ $x(t) = \frac{1}{2}t$ $p(t) = -t + C$ as befor

- For $t \in [t^*, 1]$ $u(t) = \frac{1}{C-t}$ as befor, $p(t) = -t + C$

$x(t) = t - \frac{1}{2(C-t)} + D$

$x(1) = 0$ $\Rightarrow$ $D = \frac{1}{2(C-1)} - 1$ $\Rightarrow$ $x(t) = t - \frac{1}{2(C-t)} + \frac{1}{2(C-1)} - 1$

$u(t^*) = 1$ $\Rightarrow$ $\frac{1}{C-t^*} = 1$ $C - t^* = 1$ $C = 1 + t^*$

$\Rightarrow x(t) = t - \frac{1}{2(1+t^*-t)} + \frac{1}{2t^*} - 1$

$x(t)$ continuous at t^* gives $t^* - \frac{1}{2} + \frac{1}{2t^*} - 1 = \frac{1}{2}t^*$

$\Rightarrow t^{*2} - 3t^* + 1 = 0$ $\Rightarrow t^* = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$ only the one with $-$ relevant because need to be in $[0,1]$

$\Rightarrow \boxed{t^* = \frac{3-\sqrt{5}}{2}}$ $\Rightarrow \boxed{C = \frac{5-\sqrt{5}}{2}}$

So,
$$\boxed{u(t) = \begin{cases} 1 & \text{for } t \in [0, t^*] \\ \frac{1}{\frac{5-\sqrt{5}}{2} - t} & \text{for } t \in (t^*, 1] \end{cases}} \quad \boxed{x(t) = \begin{cases} \frac{1}{2}t & \text{for } t \in [0, t^*] \\ t - \frac{1}{2\left(\frac{5-\sqrt{5}}{2} - t\right)} + \frac{1}{3-\sqrt{5}} - 1 & \text{for } t \in (t^*, 1] \end{cases}}$$

$\boxed{p(t) = -t + \frac{5-\sqrt{5}}{2}}$ $\boxed{t^* = \frac{3-\sqrt{5}}{2}}$