Seminar 2 - ECON 4140

Problem 1c Exam 08

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i) $$(x\ y\ z)\begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2y\ \ 2x+3y\ \ 0)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2xy+2xy+3y^2 = \underline{3y^2+4xy}$$

no z

can take both positive and negative values $\Rightarrow$ <u>Indefinite</u>

ii) $$\begin{pmatrix} 16 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Rightarrow \begin{pmatrix} 16x+z \\ x+2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

← irrelevant because of z

← relevant

$\Rightarrow x+2y=0$ so $x=-2y$

$$\Rightarrow (x\ y\ z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3y^2+4(-2y)y = 3y^2-8y^2 = -5y^2 < 0$$ $\Rightarrow$ <u>Negative definite</u>

Problem 1a,b Exam 09

$Q(x,y,z) = x^2+y^2+z^2+2axy+2xz+2yz$ Find Hesian!

$Q'_x = 2x+2ay+2z \qquad Q'_y = 2y+2ax+2z \qquad Q'_z = 2z+2x+2y$

$$H = \begin{pmatrix} 2 & 2a & 2 \\ 2a & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & a & 1 \\ a & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

a) $$\begin{vmatrix} 1 & a & 1 \\ a & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - a\begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} = (1-a)\begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} = (1-a)(a-1) = -(a-1)^2 \leq 0$$

$\Rightarrow$ ~~Not positive definite for all a~~ <u>Not positive definite for any a</u>

b) $a=1$ is needed for $\Delta_3 \geq 0$ $(\Delta_3 = 0)$

with $a=1$ all the principal minors equals 0 $\Rightarrow$ <u>Positive semidefinite for $a=1$</u>

the λ satisfying this ∀ x is called eigen values $(x \neq 0)$

c) ~~Eigen value $Hx = \lambda x$~~ or $(H-\lambda I)x = 0$ ← ~~only true if $|H-\lambda I| = 0$~~

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} = (1-\lambda)^3 - (1-\lambda) - (1-\lambda) + 1 + 1 - (1-\lambda) = 0$$

$\Rightarrow (1-\lambda)^3 - 3(1-\lambda) + 2 = 0$ THIS WAS NOT EXERCISE!

~~$(1-\lambda)(1-2\lambda+\lambda^2)-3(1-\lambda)+2 = 1-2\lambda+\lambda^2-\lambda+2\lambda^2-\lambda^3-3+3\lambda+2 = 0$~~

~~$-\lambda^3+3\lambda^2=0 \quad \lambda^2(\lambda-3)=0 \Rightarrow \underline{\lambda=0}$ or $\underline{\lambda=3}$~~

~~Two eigenvalues $\underline{\underline{\lambda_1=0}}$ $\underline{\underline{\lambda_2=3}}$~~

$\boxed{\text{Problem 1-07}}$

a) $A=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad A^{-1}A=I$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} x_{11}a & x_{12}b & x_{13}c \\ x_{21}a & x_{22}b & x_{23}c \\ x_{31}a & x_{32}b & x_{33}c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow x_{11}a=1$ so $\boxed{x_{11}=\frac{1}{a}}$ $\boxed{x_{12}=0}$ $\boxed{x_{13}=0}$

$\boxed{x_{21}=0}$ $x_{22}b=1$ so $\boxed{x_{22}=\frac{1}{b}}$ $\boxed{x_{23}=0}$

$\boxed{x_{31}=0}$ $\boxed{x_{32}=0}$ $x_{33}c=1$ so $\boxed{x_{33}=\frac{1}{c}}$

$$\Rightarrow \underline{\underline{A^{-1}=\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}}}$$

b) $b_i b_j=0$ for $i\neq j$ Let $b_i=(b_{1i}\ b_{2i}\ b_{3i}) \Rightarrow b_i\cdot b_j = $ ~~$b_{1i}b_{j1}+b_{12}b_{j2}+b_{13}b_{j3}$~~ $b_{1i}b_{1j}+b_{2i}b_{2j}+b_{3i}b_{3j}=0$ $(i\neq j)$

$B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \qquad B'=\begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$

$$B'B=\begin{pmatrix} b_{11}^2+b_{21}^2+b_{31}^2 & b_{11}b_{12}+b_{21}b_{22}+b_{31}b_{32} & b_{11}b_{13}+b_{21}b_{23}+b_{31}b_{33} \\ b_{12}b_{11}+b_{22}b_{21}+b_{32}b_{31} & b_{12}^2+b_{22}^2+b_{32}^2 & b_{12}b_{13}+b_{22}b_{23}+b_{32}b_{33} \\ b_{13}b_{11}+b_{23}b_{21}+b_{33}b_{31} & b_{13}b_{12}+b_{23}b_{22}+b_{33}b_{32} & b_{13}^2+b_{23}^2+b_{33}^2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} b_{11}^2+b_{21}^2+b_{31}^2 & 0 & 0 \\ 0 & b_{12}^2+b_{21}^2+b_{32}^2 & 0 \\ 0 & 0 & b_{13}^2+b_{23}^2+b_{33}^2 \end{pmatrix}}} \qquad \underline{\underline{\text{Diagonal matrix}}}$$

c) $A = B'B \Rightarrow B^{-1} = A^{-1}B' = \underline{\underline{A^{-1}B}}$

sufficient answer

since A^{-1} diagonal (see problem a)

d) $(1\ -8\ 4)\cdot(-8\ 1\ 4) = -8-8+16 = \underline{0}$ $\quad (1\ -8\ 4)\cdot(4\ 4\ 7) = 4-32+28 = \underline{0}$

$(-8\ 1\ 4)\cdot(4\ 4\ 7) = -32+4+28 = \underline{0}$ $\quad$ Mutually orthogonal

$$P^{-1} = (P'P)^{-1}P = \begin{pmatrix} \frac{1}{1^2+8^2+4^2} & 0 & 0 \\ 0 & \frac{1}{8^2+1^2+4^2} & 0 \\ 0 & 0 & \frac{1}{4^2+4^2+7^2} \end{pmatrix} \begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{1}{81} & 0 & 0 \\ 0 & \frac{1}{81} & 0 \\ 0 & 0 & \frac{7}{81} \end{pmatrix}}}$$

Problem 1-13

a) $(X+\frac{1}{2}A^{-1}B')'A(X+\frac{1}{2}A^{-1}B') - \frac{1}{4}BA^{-1}B'$

$(A')^{-1} = A^{-1}$ (A=A' since A symmetric)

$= (X' + \frac{1}{2}(A^{-1}B')')(AX+\frac{1}{2}B') - \frac{1}{4}BA^{-1}B' = (X' + \frac{1}{2}B(A^{-1})')(AX+\frac{1}{2}B') - \frac{1}{4}BA^{-1}B'$

~~$= X' + \frac{1}{2}BA^{-1}AX + \frac{1}{4}BA^{-1}B' - \frac{1}{4}BA^{-1}B' = X' + \frac{1}{2}BX$~~

$= X'AX + \frac{1}{2}X'B' + \frac{1}{2}BX + \frac{1}{4}BA^{-1}B' - \frac{1}{4}BA^{-1}B' =$ ~~$X'AX + \frac{1}{2}X'B' + \frac{1}{2}BX$~~

one number $\Rightarrow BX)' = BX$

$= X'AX + \frac{1}{2}(BX)' + \frac{1}{2}BX = \underline{\underline{X'AX + BX}}$

b) If $Y = X + \frac{1}{2}A^{-1}B' = \mathbf{0}$, then $Y'XY = 0$, this is smallest possible since

$v'Xv > 0$ if $v \neq \mathbf{0}$.

Thus, $\underline{\underline{X = -\frac{1}{2}A^{-1}B' \text{ minimizes } X'AX+BX}}$

Problem 2-04

$f(x,y) = e^{-x}\sqrt{1+y^2}$ strictly convex for $|y|<1$?

$f'_x = -f \qquad f'_y = \frac{1}{2}e^{-x}(1+y^2)^{-\frac{1}{2}}2y = e^{-x}y(1+y^2)^{-\frac{1}{2}}$

$$H = \begin{pmatrix} f & -e^{-x}y(1+y^2)^{-\frac{1}{2}} \\ -e^{-x}y(1+y^2)^{-\frac{1}{2}} & e^{-x}[(1+y^2)^{-\frac{1}{2}} - y^2(1+y^2)^{-\frac{3}{2}} \end{pmatrix}$$

$|H| = e^{-2x}[1-y^2(1+y^2)^{-1}] + e^{-2x}y^2(1+y^2)^{-1} = e^{-2x} > 0$

$|f| = e^{-x}\sqrt{1+y^2} > 0 \qquad \Rightarrow$ Strictly convex

- $g(y) = f(Ay)$ y n-vector, A m×n matrix

a) $$\frac{dg(y)}{dy_i} = \frac{df(\overset{x}{\widetilde{Ay}})}{dy_i} = \sum_{k=1}^{m} \frac{\partial f}{\partial x_k}\frac{\partial x_k}{\partial y_i}$$

← k'th element of Ay (this is m-vector)

$$Ay = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m}\\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{n} a_{1i}y_i\\ \sum_{i=1}^{n} a_{2i}y_i\\ \vdots\\ \sum_{i=1}^{n} a_{mi}y_i\end{pmatrix}$$

$\Rightarrow x_k = \sum_{i=1}^{n} a_{ki}y_i$

$$\boxed{\frac{\partial x_k}{\partial y_i} = a_{ki}}$$

REMEMBER THIS!

∇f

$$= \underbrace{\left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_m}\right)}_{\nabla f}\begin{pmatrix}\frac{\partial x_1}{\partial y_i}\\ \vdots\\ \frac{\partial x_m}{\partial y_i}\end{pmatrix}$$

← element i of the gradient

$$\frac{d^2 g(y)}{dy_i^2} = \sum_{k=1}^{m}\left[\left(\frac{\partial^2 f}{\partial x_k \partial y_i}\right)\frac{\partial x_k}{\partial y_i} + \underbrace{\frac{\partial f}{\partial x_k}\frac{\partial^2 x_k}{\partial y_i^2}}_{=0 \text{ since } Ay \text{ linear}}\right] = \sum_{k=1}^{m}\left[\left(\sum_{j=1}^{m}\frac{\partial^2 f}{\partial x_k \partial x_j}\frac{\partial x_j}{\partial y_i}\right)\frac{\partial x_k}{\partial y_i}\right]$$

H_f

$$= \left(\frac{\partial x_1}{\partial y_i} \cdots \frac{\partial x_m}{\partial y_i}\right)\begin{pmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}\\ \vdots & \ddots & & \vdots\\ & & \ddots & \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_m^2}\end{pmatrix}\begin{pmatrix}\frac{\partial x_1}{\partial y_i}\\ \frac{\partial x_2}{\partial y_i}\\ \vdots\\ \frac{\partial x_m}{\partial y_i}\end{pmatrix}$$

← element i,i of Hessian

$$\frac{dg(y)}{dy_i dy_j} = \sum_{k=1}^{m}\left[\frac{\partial^2 f}{\partial x_k \partial y_j}\frac{\partial x_k}{\partial y_i} + \underbrace{\frac{\partial f}{\partial x_k}\frac{\partial^2 x_k}{\partial y_i \partial y_j}}_{=0}\right] = \sum_{k=1}^{m}\left[\left(\sum_{l=1}^{m}\frac{\partial^2 f}{\partial x_k \partial x_l}\frac{\partial x_l}{\partial y_j}\right)\frac{\partial x_k}{\partial y_i}\right]$$

$$= \underbrace{\left(\frac{\partial x_1}{\partial y_j} \cdots \frac{\partial x_m}{\partial y_j}\right)}_{c_j^T}\begin{pmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}\\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_m^2}\end{pmatrix}\underbrace{\begin{pmatrix}\frac{\partial x_1}{\partial y_i}\\ \frac{\partial x_2}{\partial y_i}\\ \vdots\\ \frac{\partial x_m}{\partial y_i}\end{pmatrix}}_{c_i}$$

↑ element i,j of Hessian

b) $r_i^T = (a_{i1}\ a_{i2} \cdots a_{in})$, $c_i = (a_{1i}\ a_{2i} \cdots a_{mi})$

row number i — column number i

$f(c_i)$ is well defined, since c_i has dimension m, same as Ay.

$\Rightarrow z_i = f(c_i) \quad z = (f(c_1)\ f(c_2) \cdots f(c_n))$

~~$\Rightarrow z = (c_1\ c_2 \cdots c_m)$ a vector of vectors!~~

c) $h(y) = g(y) - z^T y$

$$\frac{\partial h(y)}{\partial y_i} = \frac{\partial g(y)}{\partial y_i} - f(c_i) = \underline{\underline{\nabla f \cdot c_i^T - f(c_i)}}$$

$$\frac{\partial^2 h(y)}{\partial y_i^2} = \frac{\partial^2 g}{\partial y_i^2} = \underline{\underline{c_i H_f c_i^T}}$$

$$\frac{\partial^2 h(y)}{\partial y_i \partial y_j} = \frac{\partial^2 g}{\partial y_i \partial y_j} = \underline{\underline{c_j H_f c_i^T}}$$

d) Since h is convex, any stationary point must be a local minima, hence the maximum value must be on the boundary

e) $n=2 \quad y_1, y_2 \geq 0 \quad y_1 + y_2 \leq 1$

$\Rightarrow$ Maxima exists when $y_1 = y_2 = 0$ or $y_1 + y_2 = 1$

only relevant to check $(y_1, y_2) = (0,1)$ and $(y_1, y_2) = (1,0)$ due to convexity (and $(y_1, y_2) = (0,0)$)

$$h(0) = g(0) - \begin{pmatrix} f(c_1) \\ f(c_2) \end{pmatrix} (0\ 0) = g(0) = f(0)$$

$$h(y_1, y_2) = g(y_1, y_2) - \begin{pmatrix} f(c_1) \\ f(c_2) \end{pmatrix} (y_1\ y_2) = f\left(A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) - \begin{pmatrix} f(c_1) \\ f(c_2) \end{pmatrix} (y_1\ y_2)$$

~~$= f(a_{11} y_1 + a_{12} y_2, a_{21} y_1 + a_{22} y_2) = f(a_{11}$~~

$$= f(a_{11} y_1 + a_{12} y_2, \cdots, a_{m1} y_1 + a_{m2} y_2) - f(a_{11}, \ldots, a_{m1}) y_1 - f(a_{12}, \ldots, a_{m2}) y_2$$

$\Rightarrow h(0,1) = f(a_{12}, \ldots, a_{m2}) - f(a_{12}, \ldots, a_{m2}) = 0$ and $h(1,0) = f(a_{11}, \ldots, a_{m1}) - f(a_{11}, \ldots, a_{m1}) = 0$

$\Rightarrow$ Maxima is fined in $(y_1, y_2) = (0,0)$ if $f(0) \geq 0$,

iff $f(0) \leq 0$ maxima is in $\underline{\underline{(y_1, y_2) = (1,0) \text{ and } (0,1)}}$

only if $f(0) \neq 0$

~~Since h is convex this f convex since ... ≥ 0, and $y_1 + y_2 = 1$ $\Rightarrow$ takes the role of λ in definition of convex function~~

- Jensen's inequality: ~~$f\left(\int_a^b \lambda(t) x(t)\, dt\right) \geq \int_a^b \lambda(t) f(x(t))\, dt$~~

  ≤ if convex

  $f(\lambda_1 x_1 + \dots + \lambda_m x_m) \geq \lambda_1 f(x_1) + \dots + \lambda_m f(x_m)$ $\Leftrightarrow$ if f concave

  $(\lambda_1 + \dots + \lambda_m = 1)$

Profit of ~~portfolio~~ mortgage portfolio concave (if profit concave)

⇒ profit of expected value higher than expected profit (see Jensen's inequality)

⇒ Loss is probably going to be larger.

(Not very satisfied with own answer... any suggestions? Discussion)